# Static Program Analysis

Seidl/Wilhelm/Hack: Compiler Design – Analysis and Transformation, Springer Verlag, 2012

## A Short History of Static Program Analysis

- Early high-level programming languages were implemented on very small and very slow machines.
- Compilers needed to generate executables that were extremely efficient in space and time.
- Compiler writers invented efficiency-increasing program transformations, wrongly called optimizing transformations.
- Transformations must not change the semantics of programs.
- Enabling conditions guaranteed semantics preservation.
- Enabling conditions were checked by static analysis of programs.

## Theoretical Foundations of Static Program Analysis

- Theoretical foundations for the solution of recursive equations: Kleene (1930s), Tarski (1955)
- Gary Kildall (1972) clarified the lattice-theoretic foundation of data-flow analysis.
- Patrick Cousot (1974) established the relation to the programming-language semantics.

## Static Program Analysis as a Verification Method

- Automatic method to derive invariants about program behavior, answers questions about program behavior:
  - will index always be within bounds at program point $p$?
  - will memory access at $p$ always hit the cache?
- answers of sound static analysis are correct, but approximate: don't know is a valid answer!
- analyses proved correct wrt. language semantics,

## Proposed Lectures Content:

1. Introductory example: rules-of-sign analysis
2. theoretical foundations: lattices
3. an operational semantics of the language
4. another example: constant propagation
5. relating the semantics to the analysis—correctness proofs
6. some further static analyses in compilers: Elimination of superfluous computations
   - $\rightarrow$ available expressions
   - $\rightarrow$ live variables
   - $\rightarrow$ array-bounds checks

# 1 Introduction

... in this course and in the Seidl/Wilhelm/Hack book:

a simple imperative programming language with:

- variables // registers
- $R = e;$ // assignments
- $R = M[e];$ // loads
- $M[e_1] = e_2;$ // stores
- if $(e)$ $s_1$ else $s_2$ // conditional branching
- goto $L;$ // no loops

Intermediate language into which (almost) everything can be compiled.
However, no procedures. So, only intra-procedural analyses!

# 2 Example: Rules-of-Sign Analysis

Starting Point: Questions about a program, mostly at a particular program point:

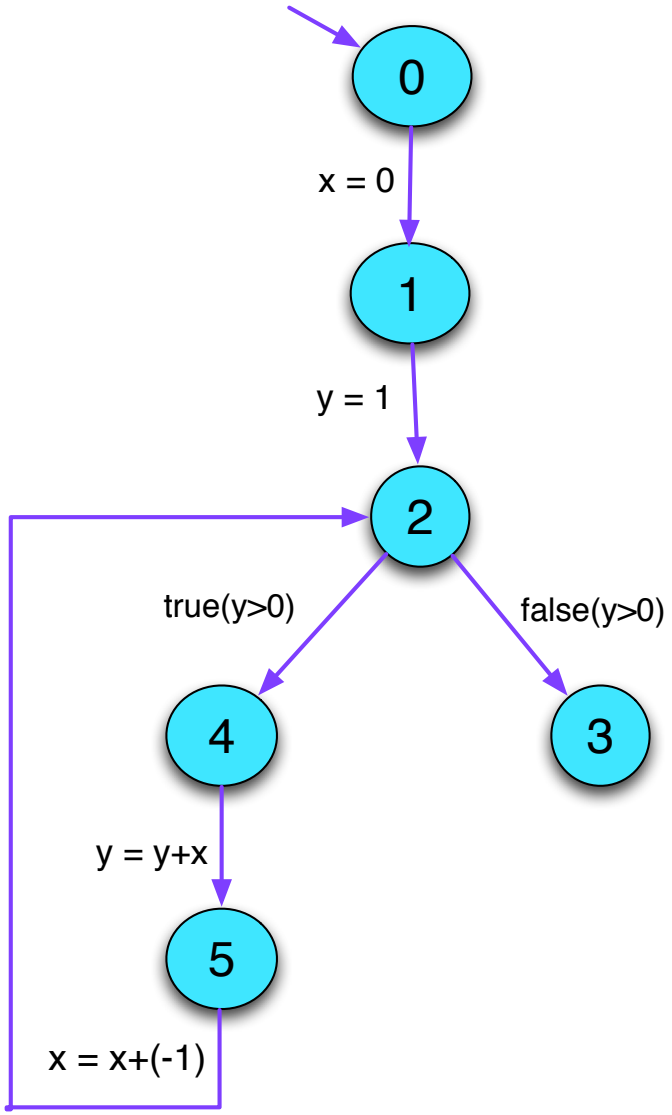
- May variable $x$ have value 0 when program execution reaches this program point? $\longrightarrow$ Attempt to exclude division by 0.
- May $x$ have a negative value? $\longrightarrow$ Attempt to exclude sqrt of a negative number.

Solution: Determine at each program point the sign of the values of all variables of numeric type.

Determines a sound, but maybe approximate answer.

Example program represented as *control-flow graph*

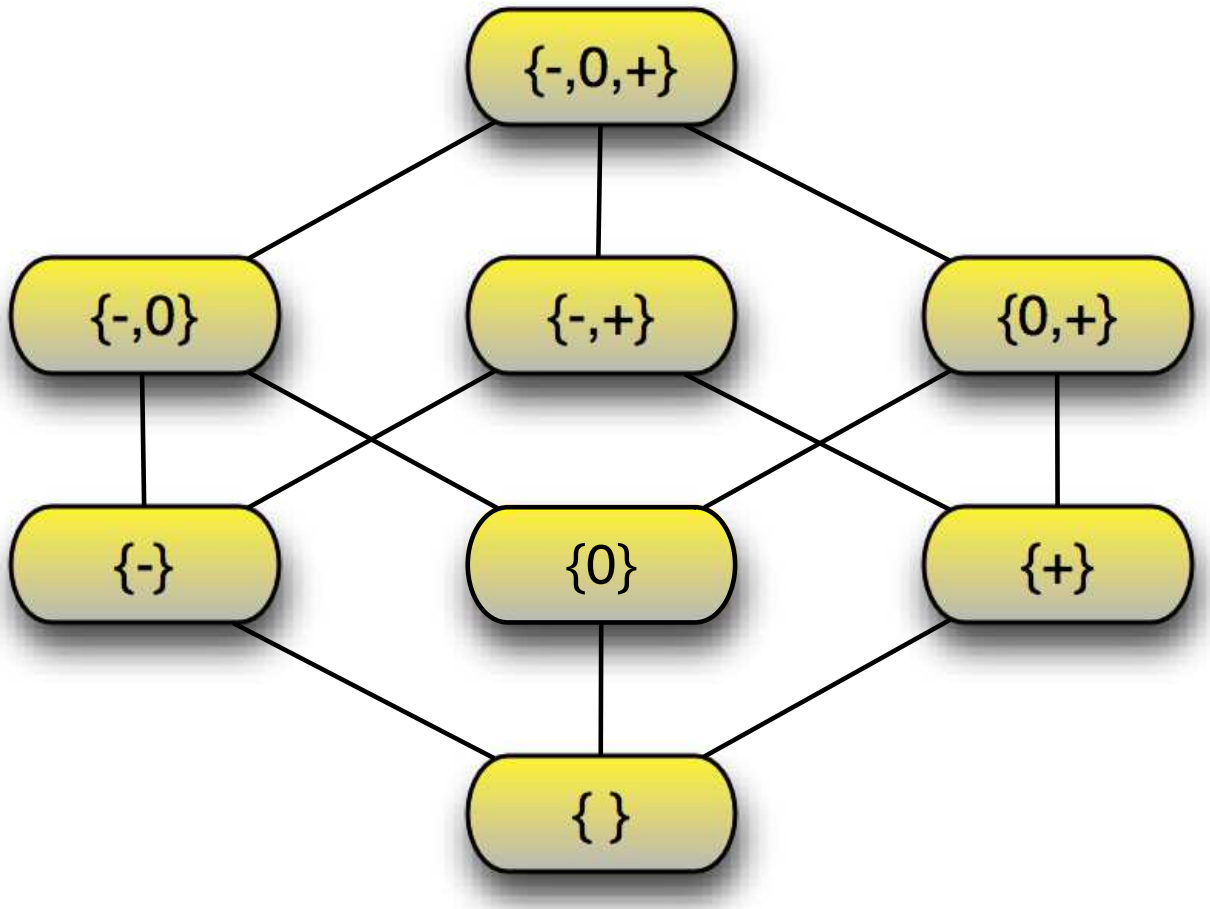
```
1: x = 0;
2: y = 1;
3: while (y > 0) do
4:       y = y + x;
5:       x = x + (-1);
```



All the ingredients:

- a set of information elements, each a set of possible signs,
- a partial order, "$\sqsubseteq$", on these elements, specifying the "relative strength" of two information elements,
- these together form the abstract domain, a lattice,
- functions describing how signs of variables change by the execution of a statement, abstract edge effects,
- these need an abstract arithmetic, an arithmetic on signs.

We construct the abstract domain for single variables starting with the lattice $Signs = 2^{\{-,0,+\}}$ with the relation "$\sqsubseteq$" ="$\subseteq$".



The analysis should ”bind” program variables to elements in $Signs$.

So, the abstract domain is $\mathbb{D} = (Vars \to Signs)_\perp$, a Sign-environment.

$\perp \in \mathbb{D}$ is the function mapping all arguments to $\{\}$.

The partial order on $\mathbb{D}$ is $D_1 \sqsubseteq D_2$ iff

$D_1 = \perp$ or

$D_1\, x \supseteq D_2\, x \quad (x \in Vars)$

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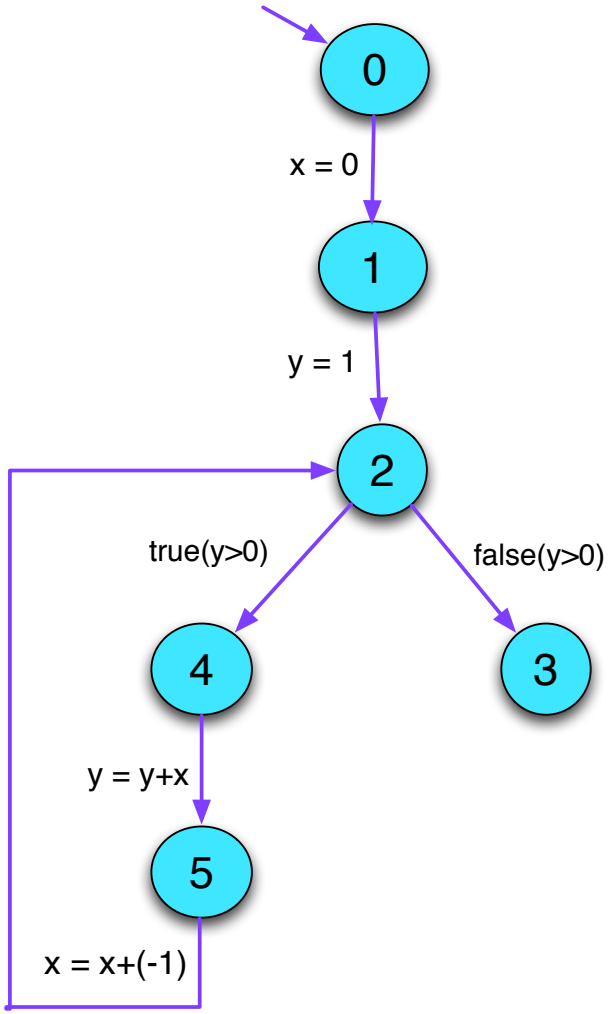
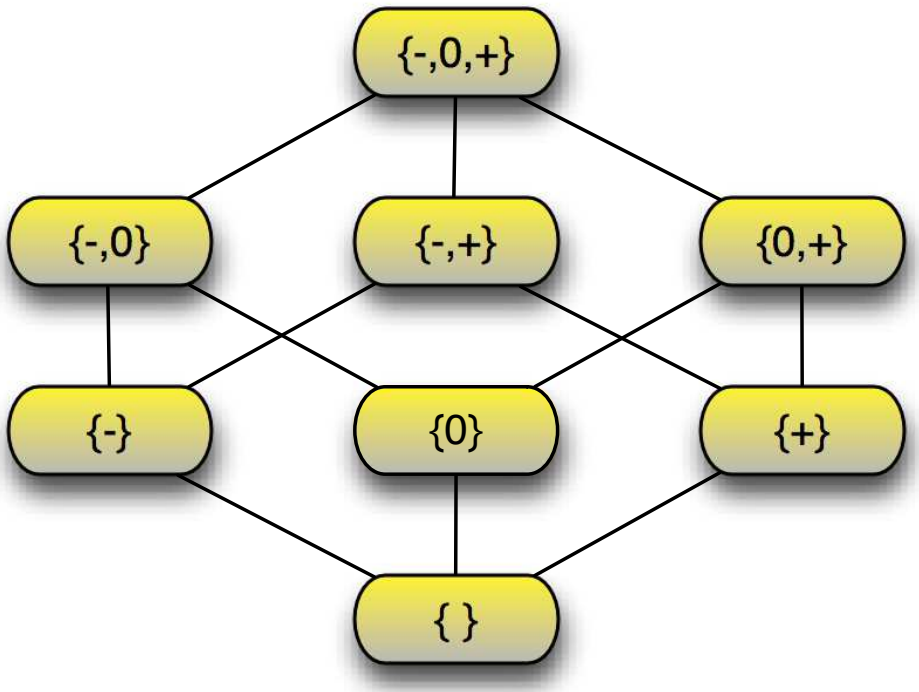
Intuition?

$D_1$ is at least as precise as $D_2$ since $D_2$ admits at least as many signs as $D_1$

How did we analyze the program?

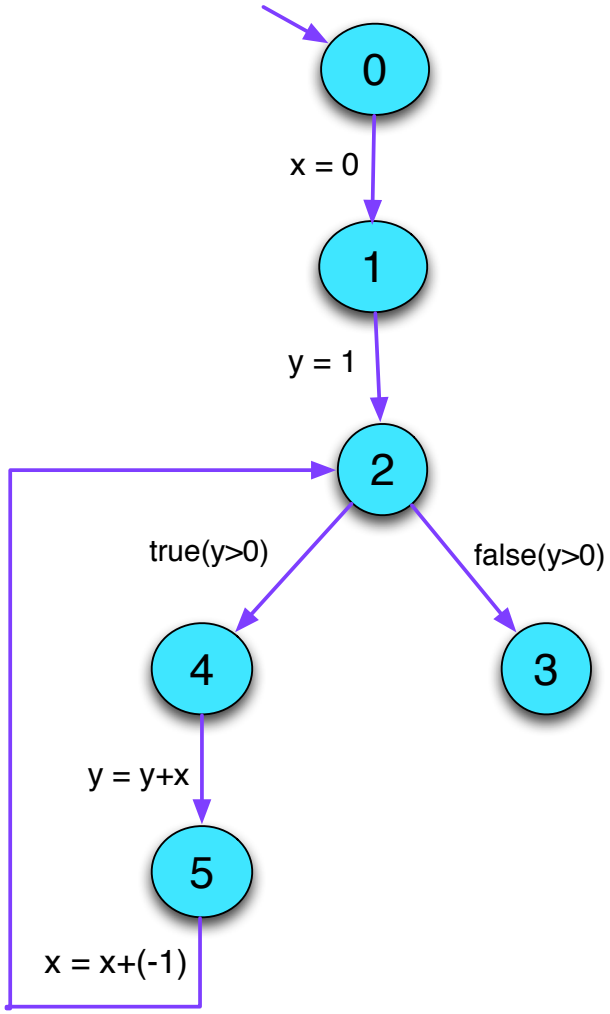



In particular, how did we walk the lattice for $y$ at program point 5?



How is a solution found?

Iterating until a fixed-point is reached




| 0 | | 1 | | 2 | | 3 | | 4 | | 5 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | $y$ | $x$ | $y$ | $x$ | $y$ | $x$ | $y$ | $x$ | $y$ | $x$ | $y$ |
| | | | | | | | | | | | |

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## Idea:

- We want to determine the signs of the values of expressions.
- For some sub-expressions, the analysis may yield $\{+, -, 0\}$, which means, it couldn't find out.
- We replace the concrete operators $\Box$ working on values by abstract operators $\Box^\sharp$ working on signs:
- The abstract operators allow to define an abstract evaluation of expressions:

$$[\![e]\!]^\sharp \; : \; (Vars \to Signs) \to Signs$$

Determining the sign of expressions in a Sign-environment is defined by the function $[\![\ ]\!] : Exp \times SignEnv \to Val$

$$
\begin{array}{lcl}
[\![c]\!]^\sharp\, D & = & \begin{cases} \{+\} & \text{if c > 0} \\ \{-\} & \text{if c < 0} \\ \{0\} & \text{if c = 0} \end{cases} \\
[\![v]\!]^\sharp & = & D(v) \\
[\![e_1 \,\square\, e_2]\!]^\sharp\, D & = & [\![e_1]\!]^\sharp\, D \,\square^\sharp\, [\![e_2]\!]^\sharp\, D \\
[\![\square e]\!]^\sharp\, D & = & \square^\sharp [\![e]\!]^\sharp\, D
\end{array}
$$

A remark about the notation:
$[\![\ ]\!]$ is given in a "distributed" form; its first argument appears between the brackets, the second follows the brackets.

Abstract operators working on signs (Addition)

| $+^{\#}$ | {0} | {+} | {-} | {-, 0} | {-, +} | {0, +} | {-, 0, +} |
|---|---|---|---|---|---|---|---|
| {0} | {0} | {+} | | | | | |
| {+} | | | | | | | |
| {-} | | | | | | | |
| {-, 0} | | | | | | | |
| {-, +} | | | | | | | |
| {0, +} | | | | | | | |
| {-, 0, +} | {-, 0, +} | | | | | | |

Abstract operators working on signs (Multiplication)

| $\times^{\#}$ | $\{0\}$ | $\{+\}$ | $\{-\}$ | $\cdots$ |
|---|---|---|---|---|
| $\{0\}$ | $\{0\}$ | $\{0\}$ | $\{0\}$ | |
| $\{+\}$ | $\{0\}$ | $\{+\}$ | $\{-\}$ | |
| $\{-\}$ | $\{0\}$ | $\{-\}$ | $\{+\}$ | |
| $\{-, 0\}$ | $\{0\}$ | $\{-, 0\}$ | $\{0, +\}$ | |
| $\{-, +\}$ | $\{0\}$ | $\{-, +\}$ | $\{-, +\}$ | |
| $\{0, +\}$ | $\{0\}$ | $\{0, +\}$ | $\{-, 0\}$ | |
| $\{-, 0, +\}$ | $\{0\}$ | $\{-, 0, +\}$ | $\{-, 0, +\}$ | |

Abstract operators working on signs (unary minus)

| $-^{\#}$ | $\{0\}$ | $\{+\}$ | $\{-\}$ | $\{-, 0\}$ | $\{-, +\}$ | $\{0, +\}$ | $\{-, 0, +\}$ |
|---|---|---|---|---|---|---|---|
| | $\{0\}$ | $\{-\}$ | $\{+\}$ | $\{+, 0\}$ | $\{-, +\}$ | $\{0, -\}$ | $\{-, 0, +\}$ |

## Working an example: $D = \{x \mapsto \{+\}, y \mapsto \{+\}\}$

$$
\begin{array}{lcl}
[\![x + 7]\!]^\sharp\, D & = & [\![x]\!]^\sharp\, D \;+^\sharp\; [\![7]\!]^\sharp\, D \\
& = & \{+\} \;+^\sharp\; \{+\} \\
& = & \{+\} \\
[\![x + (-y)]\!]^\sharp\, D & = & \{+\} \;+^\sharp\; (-^\sharp [\![y]\!]^\sharp\, D\,) \\
& = & \{+\} \;+^\sharp\; (-^\sharp \{+\}) \\
& = & \{+\} \;+^\sharp\; \{-\} \\
& = & \{+, -, 0\}
\end{array}
$$

$[\![lab]\!]^\sharp$ is the abstract edge effects associated with edge $k$.

It depends only on the label *lab*:

$$
\begin{array}{lcl}
[\![;]\!]^\sharp\ D & = & D \\
[\![\mathrm{true}\,(e)]\!]^\sharp\ D & = & D \\
[\![\mathrm{false}\,(e)]\!]^\sharp\ D & = & D \\
[\![x = e;]\!]^\sharp\ D & = & D \oplus \{x \mapsto [\![e]\!]^\sharp\ D\} \\
[\![x = M[e];]\!]^\sharp\ D & = & D \oplus \{x \mapsto \{+, -, 0\}\} \\
[\![M[e_1] = e_2;]\!]^\sharp\ D & = & D
\end{array}
$$

... whenever $D \neq \bot$

These edge effects can be composed to the effect of a path $\pi = k_1 \ldots k_r$:

$$[\![\pi]\!]^\sharp = [\![k_r]\!]^\sharp \circ \ldots \circ [\![k_1]\!]^\sharp$$

Consider a program node $v$:

$\rightarrow$ For every path $\pi$ from program entry *start* to $v$ the analysis should determine for each program variable $x$ the set of all signs that the values of $x$ may have at $v$ as a result of executing $\pi$.

$\rightarrow$ Initially at program start, no information about signs is available.

$\rightarrow$ The analysis computes a superset of the set of signs as safe information.

$\Longrightarrow$ For each node $v$, we need the set:

$$\mathcal{S}[v] \quad = \quad \bigcup \{ [\![\pi]\!]^\sharp \top \mid \pi : start \rightarrow^* v \}$$

where $\top$ is the function bindig all variables to $\{-, 0, +\}$.

This function describes that we don't know the sign of any variable at program entry.

## Question:

How do we compute $\mathcal{S}[u]$ for every program point $u$?

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Collect all constraints on the values of $\mathcal{S}[u]$ into a system of constraints:

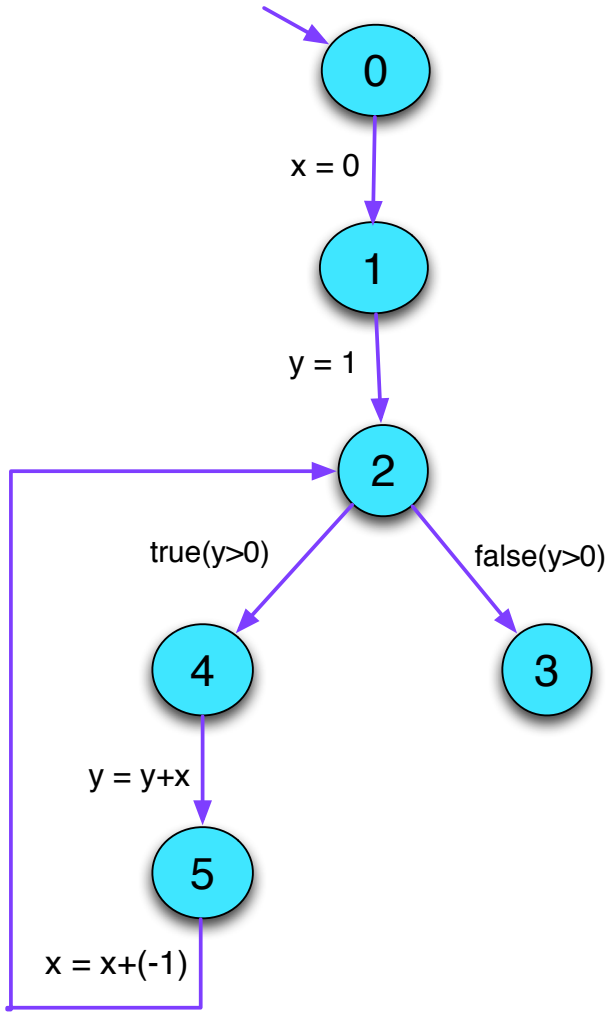
$$\begin{array}{lcll} \mathcal{S}[start] & \supseteq & \top & \\ \mathcal{S}[v] & \supseteq & [\![k]\!]^\sharp\,(\mathcal{S}[u]) & k = (u, \_, v) \quad \text{edge} \end{array}$$

Why $\supseteq$?

## Wanted:

- a least solution (why least?)
- an algorithm that computes this solution

## Example:



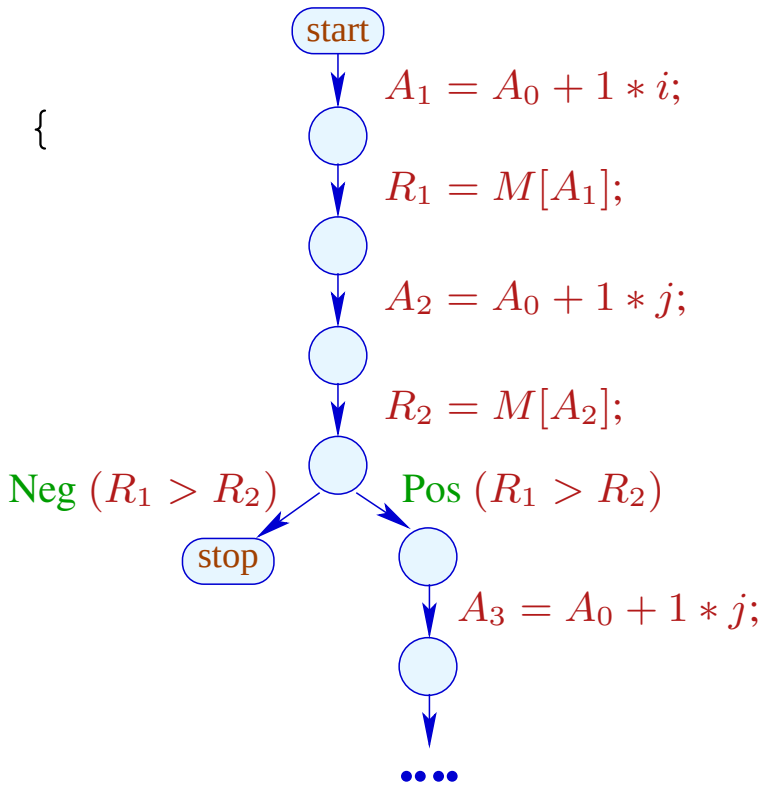

$$
\begin{aligned}
\mathcal{S}[0] &\supseteq \top \\
\mathcal{S}[1] &\supseteq \mathcal{S}[0] \oplus \{x \mapsto \{0\}\} \\
\mathcal{S}[2] &\supseteq \mathcal{S}[1] \oplus \{y \mapsto \{+\}\} \\
\mathcal{S}[2] &\supseteq \mathcal{S}[5] \oplus \{x \mapsto [\![x + (-1)]\!]^\sharp \, \mathcal{S}[5]\} \\
\mathcal{S}[3] &\supseteq \mathcal{S}[2] \\
\mathcal{S}[4] &\supseteq \mathcal{S}[2] \\
\mathcal{S}[5] &\supseteq \mathcal{S}[4] \oplus \{y \mapsto [\![y + x]\!]^\sharp \, \mathcal{S}[4]\}
\end{aligned}
$$

# 3 An Operational Semantics

Programs are represented as control-flow graphs.

Example:

```
void swap (int i, int j) {
      int t;
      if (a[i] > a[j]) {
          t = a[j];
          a[j] = a[i];
          a[i] = t;
        }
      }
```



Thereby, represent:

| | |
|---|---|
| vertex | program point |
| start | program start |
| stop | program exit |
| edge | labeled with a statement or a condition |

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| vertex | program point |
|---|---|
| start | program start |
| stop | program exit |
| edge | step of computation |

## Edge Labelings:

**Test** : Pos $(e)$ or Neg $(e)$ (better true$(e)$ or false$(e)$)

**Assignment** : $R = e;$

**Load** : $R = M[e];$

**Store** : $M[e_1] = e_2;$

**Nop** : ;

Execution of a path is a computation.

A computation transforms a state $s = (\rho, \mu)$ where:

| | |
|---|---|
| $\rho : \mathit{Vars} \to \mathbf{int}$ | values of variables (contents of symbolic registers) |
| $\mu : \mathbb{N} \to \mathbf{int}$ | contents of memory |

Every edge $k = (u, lab, v)$ defines a partial transformation

$$[\![k]\!] = [\![lab]\!]$$

of the state:

$$
\begin{array}{lcll}
[\![;]\!]\,(\rho,\mu) & = & (\rho,\mu) & \\
 & & & \\
[\![\mathrm{true}\,(e)]\!]\,(\rho,\mu) & = & (\rho,\mu) & \text{if } [\![e]\!]\,\rho \neq 0 \\
[\![\mathrm{false}\,(e)]\!]\,(\rho,\mu) & = & (\rho,\mu) & \text{if } [\![e]\!]\,\rho = 0
\end{array}
$$

$$[\![;]\!]\,(\rho,\mu) \quad = \quad (\rho,\mu)$$

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// $[\![e]\!]$ : evaluation of the expression $e$, e.g.

// $[\![x+y]\!]\,\{x \mapsto 7, y \mapsto -1\} = 6$

// $[\![!(x == 4)]\!]\,\{x \mapsto 5\} = 1$

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$$[\![R = e;]\!]\,(\rho,\mu) \quad = \quad (\boxed{\rho \oplus \{R \mapsto [\![e]\!]\,\rho\}}, \mu)$$

// where "$\oplus$" modifies a mapping at a given argument

$$[\![R = M[e];]\!]\,(\rho,\mu) \quad = \quad (\boxed{\rho \oplus \{R \mapsto \mu([\![e]\!]\,\rho)\}}, \mu)$$

$$[\![M[e_1] = e_2;]\!]\,(\rho,\mu) \quad = \quad (\rho, \boxed{\mu \oplus \{[\![e_1]\!]\,\rho \mapsto [\![e_2]\!]\,\rho\}})$$

## Example:

$[\![x = x + 1;]\!]\,(\{x \mapsto 5\}, \mu) = (\rho, \mu)$ where

$$\begin{aligned} \rho &= \{x \mapsto 5\} \oplus \{x \mapsto [\![x+1]\!]\,\{x \mapsto 5\}\} \\ &= \{x \mapsto 5\} \oplus \{x \mapsto 6\} \\ &= \{x \mapsto 6\} \end{aligned}$$

A path $\pi = k_1 k_2 \ldots k_m$ defines a computation in the state s if

$$s \in def\,([\![k_m]\!] \circ \ldots \circ [\![k_1]\!])$$

The result of the computation is $[\![\pi]\!]\,s = ([\![k_m]\!] \circ \ldots \circ [\![k_1]\!])\,s$

## The approach:

A static analysis needs to collect correct and hopefully precise information about a program in a terminating computation.

Concepts:

- partial orders relate information for their contents/quality/precision,
- least upper bounds combine information in the best possible way,
- monotonic functions preserve the order, prevent loss of collected information, prevent oscillation.

# 4 Complete Lattices

A set $\mathbb{D}$ together with a relation $\sqsubseteq \subseteq \mathbb{D} \times \mathbb{D}$ is a partial order if for all $a, b, c \in \mathbb{D}$,

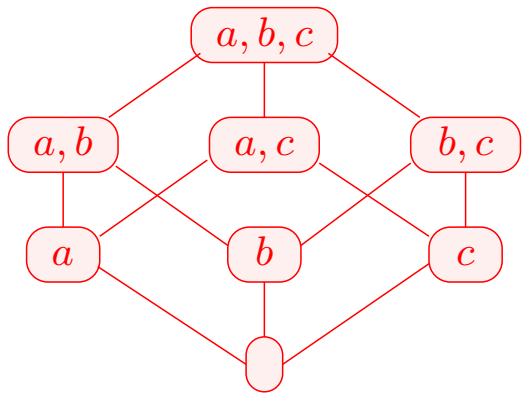
$$
\begin{array}{lll}
a \sqsubseteq a & & \textit{reflexivity} \\
a \sqsubseteq b \wedge b \sqsubseteq a & \Longrightarrow \; a = b & \textit{anti-symmetry} \\
a \sqsubseteq b \wedge b \sqsubseteq c & \Longrightarrow \; a \sqsubseteq c & \textit{transitivity}
\end{array}
$$

Intuition: $\sqsubseteq$ represents precision.

By convention: $a \sqsubseteq b$ means $a$ is at least as precise as $b$.

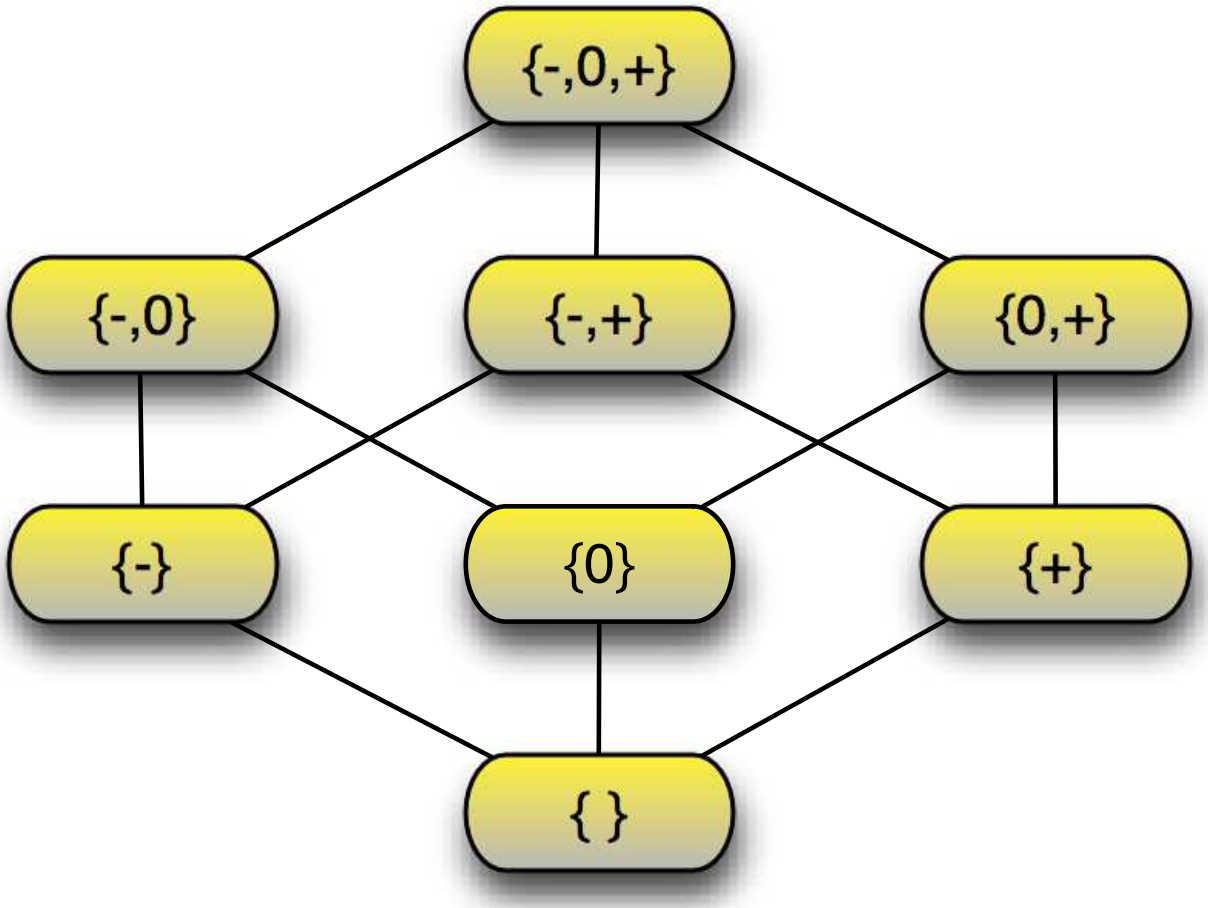
## Examples:

1. $\mathbb{D} = 2^{\{a,b,c\}}$ with the relation "$\subseteq$" :



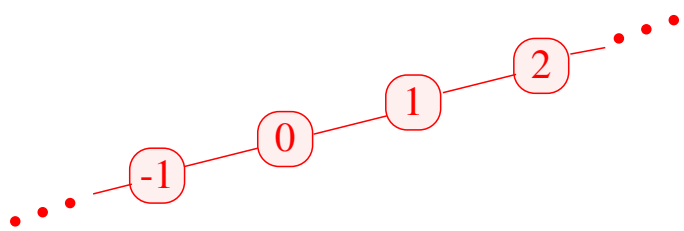
## Examples:

1. The rules-of-sign analysis uses the following lattice $\mathbb{D} = 2^{\{-,0,+\}}$ with the relation "$\subseteq$" :

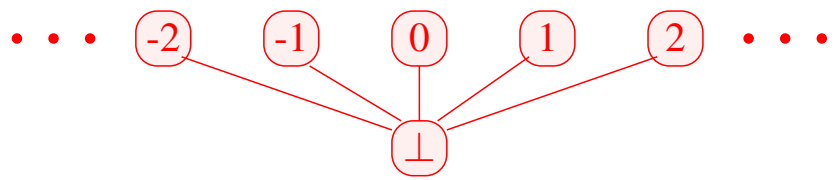


2. $\mathbb{Z}$ with the relation "$=$" :


2. $\mathbb{Z}$ with the relation "$\leq$" :




3. $\mathbb{Z}_\perp = \mathbb{Z} \cup \{\perp\}$ with the ordering:



$d \in \mathbb{D}$ is called upper bound for $X \subseteq \mathbb{D}$ if

$$x \sqsubseteq d \qquad \text{for all } x \in X$$

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The least upper bound is the youngest common ancestor in the pictorial representation of lattices.

Intuition: It is the best combined information for $X$.

## Caveat:

- $\{0, 2, 4, \ldots\} \subseteq \mathbb{Z}$ has no upper bound!
- $\{0, 2, 4\} \subseteq \mathbb{Z}$ has the upper bounds $4, 5, 6, \ldots$

A partially ordered set $\mathbb{D}$ is a complete lattice (cl) if every subset $X \subseteq \mathbb{D}$ has a least upper bound $\bigsqcup X \in \mathbb{D}$.

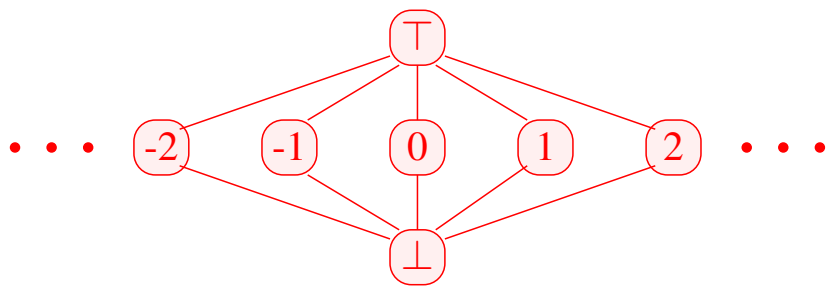
## Note:

Every complete lattice has

- $\rightarrow$ a least element $\bot = \bigsqcup \emptyset \in \mathbb{D}$;
- $\rightarrow$ a greatest element $\top = \bigsqcup \mathbb{D} \in \mathbb{D}$.

## Examples:

1. $\mathbb{D} = 2^{\{-,0,+\}}$ with $\sqsubseteq$ is a complete lattice
2. $\mathbb{D} = \mathbb{Z}$ with "$\leq$" is not a complete lattice.
3. $\mathbb{D} = \mathbb{Z}_\perp$ is also not a complete lattice
4. With an extra element $\top$, we obtain the flat lattice $\mathbb{Z}_\perp^\top = \mathbb{Z} \cup \{\perp, \top\}$ :



## Theorem:

If $\mathbb{D}$ is a complete lattice, then every subset $X \subseteq \mathbb{D}$ has a greatest lower bound $\bigsqcap X$.

Back to the system of constraints for Rules-of-Signs Analysis!

$$\begin{array}{lcll} \mathcal{S}[start] & \sqsupseteq & \top & \\ \mathcal{S}[v] & \sqsupseteq & [\![k]\!]^\sharp\,(\mathcal{S}[u]) & k = (u, \_, v) \quad \text{edge} \end{array}$$

Combine all constraints for a variable $v$ by least-upper-bound operator $\bigsqcup$:

$$\mathcal{S}[v] \quad \sqsupseteq \quad \bigsqcup\{[\![k]\!]^\sharp\,(\mathcal{S}[u]) \mid k = (u, \_, v) \ \text{ edge}\}$$

Our generic form of the systems of constraints:

$$x_i \quad \sqsupseteq \quad f_i(x_1, \ldots, x_n) \qquad\qquad (*)$$

Relation to the running example:

| | | | |
|---|---|---|---|
| $x_i$ | unknown | here: | $\mathcal{S}[u]$ |
| $\mathbb{D}$ | values | here: | $Signs$ |
| $\sqsubseteq \ \subseteq \ \mathbb{D} \times \mathbb{D}$ | ordering relation | here: | $\subseteq$ |
| $f_i\colon \mathbb{D}^n \to \mathbb{D}$ | constraint | here: | ... |

A mapping $f : \mathbb{D}_1 \to \mathbb{D}_2$ is called monotonic (order preserving) if $f(a) \sqsubseteq f(b)$ for all $a \sqsubseteq b$.

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(2) $\mathbb{D}_1 = \mathbb{D}_2 = \mathbb{Z}$ (with the ordering "$\leq$"). Then:

- $\mathsf{inc}\, x = x + 1$ is monotonic.
- $\mathsf{dec}\, x = x - 1$ is monotonic.

A mapping $f : \mathbb{D}_1 \to \mathbb{D}_2$ is called monotonic (order preserving) if $f(a) \sqsubseteq f(b)$ for all $a \sqsubseteq b$.

## Examples:

(1) $\mathbb{D}_1 = \mathbb{D}_2 = 2^U$ for a set $U$ and $f\, x = (x \cap a) \cup b$.

Obviously, every such $f$ is monotonic

(2) $\mathbb{D}_1 = \mathbb{D}_2 = \mathbb{Z}$ (with the ordering "$\leq$"). Then:

- $\mathsf{inc}\, x = x + 1$ is monotonic.
- $\mathsf{dec}\, x = x - 1$ is monotonic.
- $\mathsf{inv}\, x = -x$ is not monotonic

## Theorem:

If $f_1 : \mathbb{D}_1 \to \mathbb{D}_2$ and $f_2 : \mathbb{D}_2 \to \mathbb{D}_3$ are monotonic, then also $f_2 \circ f_1 : \mathbb{D}_1 \to \mathbb{D}_3$

## Theorem:

If $\mathbb{D}$ is a complete lattice, then the set $[S \to \mathbb{D}]$ of functions $f : S \to \mathbb{D}$ is also a complete lattice where

$$f \sqsubseteq g \quad \text{iff} \quad f\,x \sqsubseteq g\,x \quad \text{for all } x \in \mathbb{D}_1$$

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In particular for $F \subseteq [S \to \mathbb{D}_2]$,

$$\bigsqcup F = f \quad \text{with} \quad f\,x = \bigsqcup \{g\,x \mid g \in F\}$$

Wanted: least solution for:

$$x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n \qquad (*)$$

where all $f_i : \mathbb{D}^n \to \mathbb{D}$ are monotonic.

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**Idea:**

- Consider $F : \mathbb{D}^n \to \mathbb{D}^n$ where

$$F(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \quad \text{with} \quad y_i = f_i(x_1, \ldots, x_n).$$

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- If all $f_i$ are monotonic, then also $F$
- We successively approximate a solution from below. We construct:

$$\underline{\bot}, \quad F\,\underline{\bot}, \quad F^2\,\underline{\bot}, \quad F^3\,\underline{\bot}, \quad \ldots$$

Intuition: This iteration eliminates unjustified assumptions.

Hope: We eventually reach a solution!

## Theorem

- $\underline{\bot}, F\,\underline{\bot}, F^2\,\underline{\bot}, \ldots$ form an ascending chain :

$$\underline{\bot} \quad \sqsubseteq \quad F\,\underline{\bot} \quad \sqsubseteq \quad F^2\,\underline{\bot} \quad \sqsubseteq \quad \ldots$$

- If $F^k\,\underline{\bot} = F^{k+1}\,\underline{\bot}$, $F^k$ is the least solution.
- If all ascending chains are finite, such a $k$ always exists.

## Proof

The first claim follows by induction:

**Foundation:** $F^0\,\underline{\bot} = \underline{\bot} \sqsubseteq F^1\,\underline{\bot}$

**Step:** Assume $F^{i-1}\,\underline{\bot} \sqsubseteq F^i\,\underline{\bot}$. Then

$$F^i\,\underline{\bot} = F\,(F^{i-1}\,\underline{\bot}) \sqsubseteq F\,(F^i\,\underline{\bot}) = F^{i+1}\,\underline{\bot}$$

since $F$ monotonic

**Step:** Assume $F^{i-1}\,\underline{\perp} \sqsubseteq F^{i}\,\underline{\perp}$. Then

$$F^{i}\,\underline{\perp} = F\,(F^{i-1}\,\underline{\perp}) \sqsubseteq F\,(F^{i}\,\underline{\perp}) = F^{i+1}\,\underline{\perp}$$

since $F$ monotonic

## Conclusion:

If $\mathbb{D}$ is finite, a solution can be found that is definitely the least solution.

## Question:

What, if $\mathbb{D}$ is not finite?

## Theorem — Knaster – Tarski

Assume $\mathbb{D}$ is a complete lattice. Then every monotonic function $f : \mathbb{D} \to \mathbb{D}$ has a least fixed point $d_0 \in \mathbb{D}$.

## Remark:

The least fixed point $d_0$ is in $P$ and a lower bound

$\Longrightarrow$ $d_0$ is the least value $x$

## Application:

Assume $$x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n \qquad (*)$$

is a system of constraints where all $f_i : \mathbb{D}^n \to \mathbb{D}$ are monotonic.

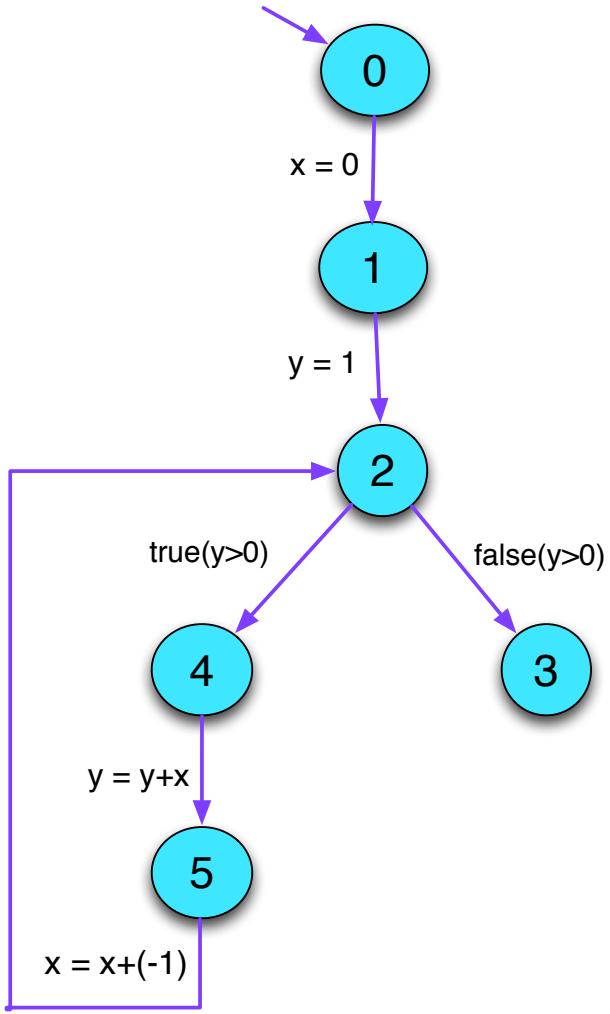
$\Longrightarrow$ least solution of $(*)$ $==$ least fixed point of $F$

## Conclusion:

Systems of inequalities can be solved through fixed-point iteration, i.e., by repeated evaluation of right-hand sides

Caveat: Naive fixed-point iteration is rather inefficient

Example:

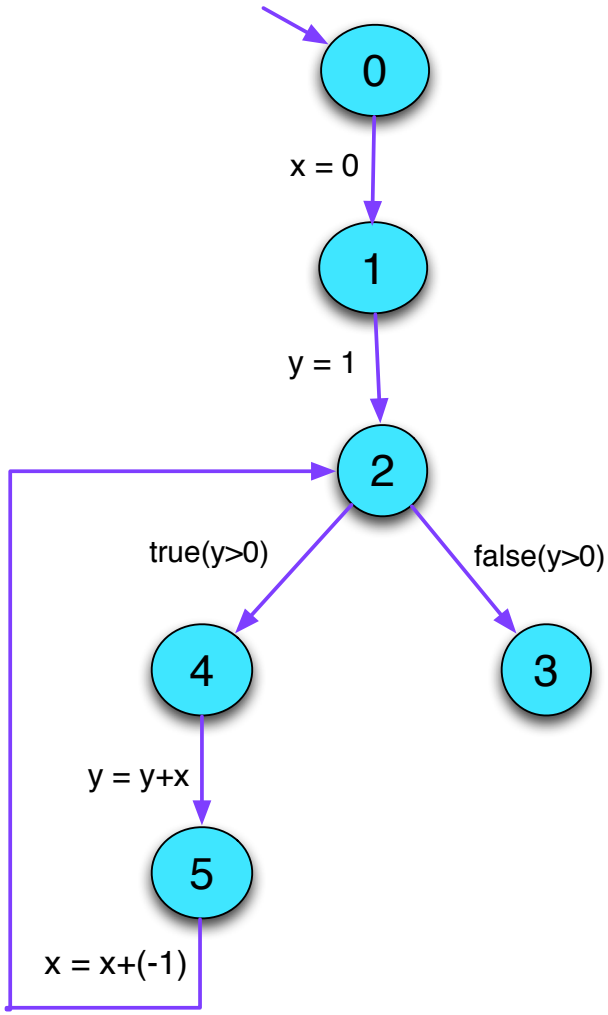



| 0 | | 1 | | 2 | | 3 | | 4 | | 5 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | $y$ | $x$ | $y$ | $x$ | $y$ | $x$ | $y$ | $x$ | $y$ | $x$ | $y$ |
| | | | | | | | | | | | |
| | | | | | | | | | | | |
| | | | | | | | | | | | |
| | | | | | | | | | | | |

## Idea: Round Robin Iteration

Instead of accessing the values of the last iteration, always use the current values of unknowns

## Example:




| 0 | | 1 | | 2 | | 3 | | 4 | | 5 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | $y$ | $x$ | $y$ | $x$ | $y$ | $x$ | $y$ | $x$ | $y$ | $x$ | $y$ |
| | | | | | | | | | | | |

The code for Round Robin Iteration in Java looks as follows:

```
for (i = 1; i ≤ n; i++) x_i = ⊥;
do {
        finished = true;
        for (i = 1; i ≤ n; i++) {
                new = f_i(x_1, ..., x_n);
                if (!(x_i ⊒ new)) {
                        finished = false;
                        x_i = x_i ⊔ new;
                }
        }
} while (!finished);
```

## What we have learned:

- The information derived by static program analysis is partially ordered in a complete lattice.
- the partial order represents information content/precision of the lattice elements.
- least upper-bound combines information in the best possible way.
- Monotone functions prevent loss of information.

For a complete lattice $\mathbb{D}$, consider systems:

$$\begin{array}{lll} \mathcal{I}[start] & \sqsupseteq & d_0 \\ \mathcal{I}[v] & \sqsupseteq & [\![k]\!]^\sharp\,(\mathcal{I}[u]) \qquad k = (u, \_, v) \quad \text{edge} \end{array}$$

where $d_0 \in \mathbb{D}$ and all $[\![k]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ are monotonic ...

Wanted: MOP (Merge Over all Paths)

$$\mathcal{I}^*[v] = \bigsqcup \{ [\![\pi]\!]^\sharp\, d_0 \mid \pi : start \to^* v \}$$

## Theorem

Kam, Ullman 1975

Assume $\mathcal{I}$ is a solution of the constraint system. Then:

$$\mathcal{I}[v] \sqsupseteq \mathcal{I}^*[v] \qquad \text{for every} \quad v$$

In particular: $\mathcal{I}[v] \sqsupseteq [\![\pi]\!]^\sharp\, d_0 \qquad$ for every $\quad \pi : start \to^* v$

Disappointment: Are solutions of the constraint system just upper bounds?

Answer: In general: yes

Notable exception, if all functions $[\![k]\!]^\sharp$ are distributive.

The function $f : \mathbb{D}_1 \to \mathbb{D}_2$ is called distributive, if
$f\,(\bigsqcup X) = \bigsqcup\{f\,x \mid x \in X\}$ for all $\emptyset \neq X \subseteq \mathbb{D}$;

Remark: If $f : \mathbb{D}_1 \to \mathbb{D}_2$ is distributive, then it is also monotonic

Theorem Kildall 1972

Assume all $v$ are reachable from *start*.

Then: If all effects of edges $[\![k]\!]^\sharp$ are distributive, $\mathcal{I}^*[v] = \mathcal{I}[v]$ holds for all $v$.

Question: Are the edge effects of the Rules-of-Sign analysis distributive?

# 5 Constant Propagation

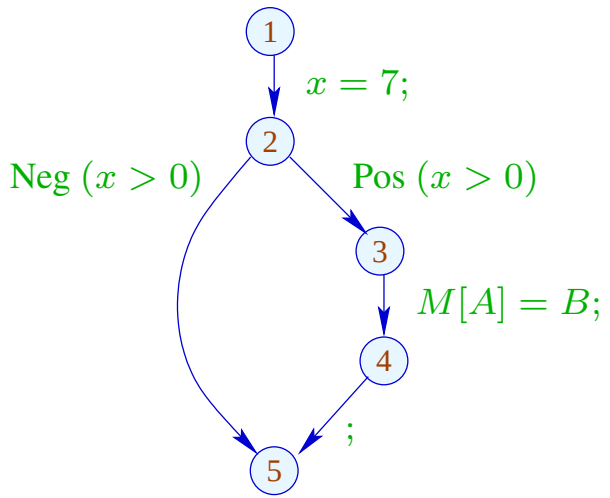
Goal: Execute as much of the code at compile-time as possible!

Example:

$x = 7;$

if $(x > 0)$

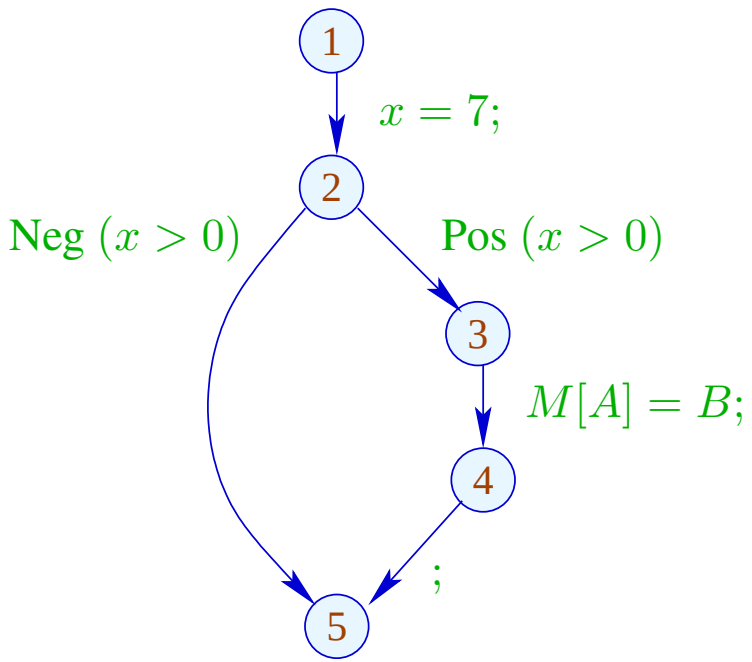
$\quad M[A] = B;$



Obviously, $x$ has always the value 7

Thus, the memory access is always executed

## Goal:



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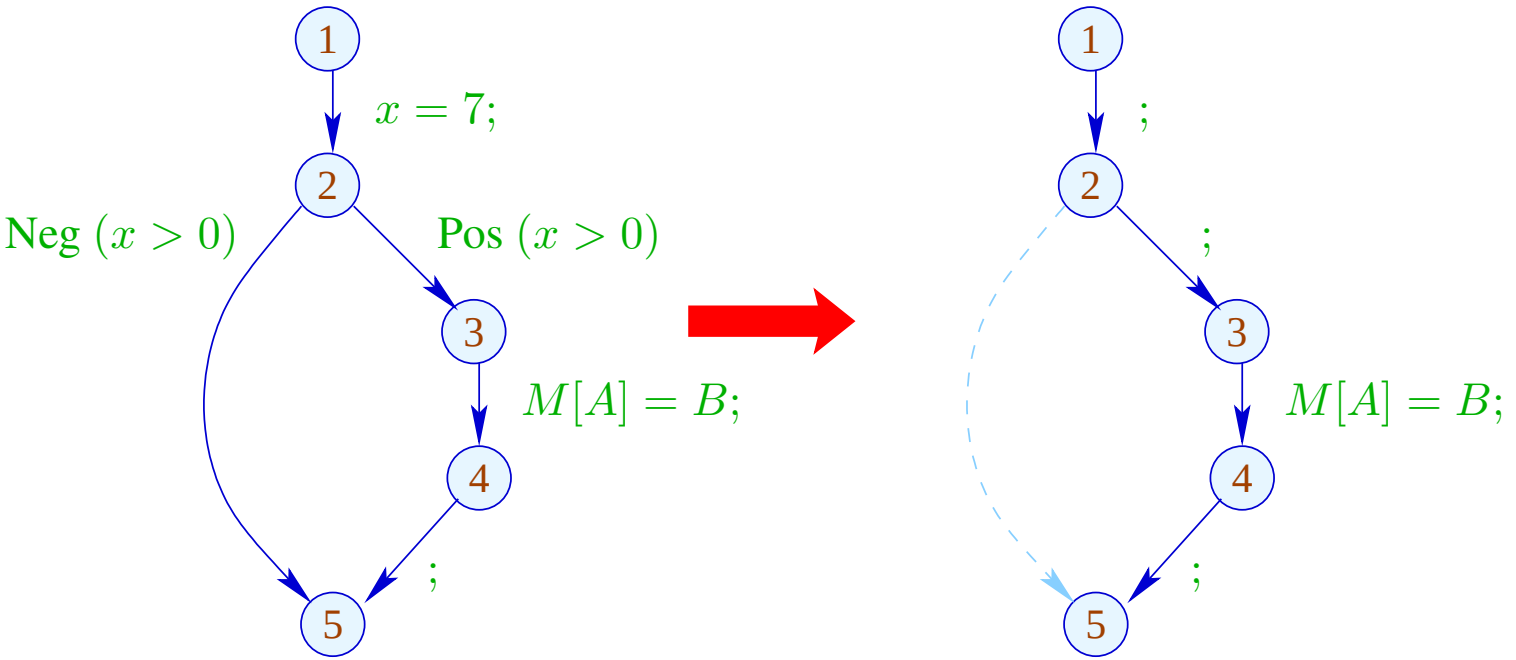

## Idea:

Design an analysis that for every program point $u$ determines the values that variables definitely have at $u$;

As a side effect, it also tells whether $u$ can be reached at all

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Design an analysis that for every program point $u$, determines the values that variables definitely have at $u$;

As a side effect, it also tells whether $u$ can be reached at all

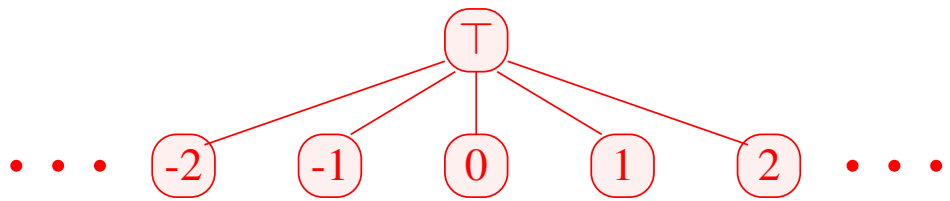
We need to design a complete lattice and an abstract semantics for this analysis.

It abstracts from the variable binding of the state, $\rho : Vars \to \mathbf{int}$, in a similar way as the Rules-of-Sign analysis.

As in the case of the Rules-of-Signs analysis the complete lattice is constructed in two steps.

(1) The potential values of variables:

$$\mathbb{Z}^{\top} = \mathbb{Z} \cup \{\top\} \qquad \text{with} \quad x \sqsubseteq y \quad \text{iff } y = \top \text{ or } x = y$$



Caveat: $\mathbb{Z}^\top$ is not a complete lattice in itself

(2) $\mathbb{D} = (\mathit{Vars} \to \mathbb{Z}^\top)_\bot = (\mathit{Vars} \to \mathbb{Z}^\top) \cup \{\bot\}$

// $\bot$ denotes: "not reachable"

with $D_1 \sqsubseteq D_2$ iff $\bot = D_1$ or

$D_1\, x \sqsubseteq D_2\, x \quad (x \in \mathit{Vars})$

Remark: $\mathbb{D}$ is a complete lattice

For every edge $k = (\_, lab, \_)$ , construct an effect function $[\![k]\!]^\sharp = [\![lab]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ which simulates the concrete computation.

Obviously, $[\![lab]\!]^\sharp \bot = \bot$ for all $lab$

Now let $\bot \neq D \in Vars \to \mathbb{Z}^\top$.

## Idea:

- We use $D$ to determine the values of expressions.

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$\Longrightarrow$

As in the Rules-of-Sign analysis, we replace the concrete operators $\Box$ by abstract operators $\Box^\sharp$ that can handle $\top$ :

$$a \,\Box^\sharp\, b = \begin{cases} \top & \text{if} \quad a = \top \text{ or } b = \top \\ a \,\Box\, b & \text{otherwise} \end{cases}$$

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$$a \,\Box^\sharp\, b = \begin{cases} \top & \text{if} \quad a = \top \text{ or } b = \top \\ a \,\Box\, b & \text{otherwise} \end{cases}$$

- The abstract operators allow to define an abstract evaluation of expressions:

$$[\![e]\!]^\sharp \;:\; (\mathit{Vars} \to \mathbb{Z}^\top) \to \mathbb{Z}^\top$$

Abstract evaluation of expressions is like the concrete evaluation — but with abstract values and operators. Here:

$$\begin{aligned} [\![c]\!]^\sharp\, D &= c \\ [\![e_1 \,\square\, e_2]\!]^\sharp\, D &= [\![e_1]\!]^\sharp\, D \,\square^\sharp\, [\![e_2]\!]^\sharp\, D \end{aligned}$$

... analogously for unary operators

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... analogously for unary operators

Example: $D = \{x \mapsto 2, y \mapsto \top\}$

$$\begin{aligned} [\![x + 7]\!]^\sharp\, D &= [\![x]\!]^\sharp\, D \;+^\sharp\; [\![7]\!]^\sharp\, D \\ &= 2 \;+^\sharp\; 7 \\ &= 9 \end{aligned}$$

$$\begin{aligned} [\![x - y]\!]^\sharp\, D &= 2 \;-^\sharp\; \top \\ &= \top \end{aligned}$$

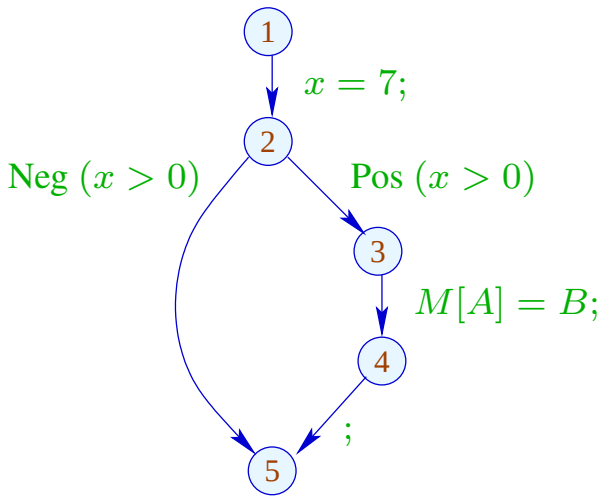
Thus, we obtain the following abstract edge effects $[\![lab]\!]^\sharp$ :

$$
\begin{array}{lcl}
[\![;]\!]^\sharp\ D & = & D \\
[\![\mathrm{true}\,(e)]\!]^\sharp\ D & = & \begin{cases} \bot & \text{if} \quad 0 = [\![e]\!]^\sharp\ D \quad \text{definitely false} \\ D & \text{otherwise} \qquad\quad \text{possibly true} \end{cases} \\
[\![\mathrm{false}\,(e)]\!]^\sharp\ D & = & \begin{cases} D & \text{if} \quad 0 \sqsubseteq [\![e]\!]^\sharp\ D \quad \text{possibly false} \\ \bot & \text{otherwise} \qquad\quad \text{definitely true} \end{cases} \\
[\![x = e;]\!]^\sharp\ D & = & D \oplus \{x \mapsto [\![e]\!]^\sharp\ D\} \\
[\![x = M[e];]\!]^\sharp\ D & = & D \oplus \{x \mapsto \top\} \\
[\![M[e_1] = e_2;]\!]^\sharp\ D & = & D
\end{array}
$$

... whenever $D \neq \bot$

At *start*, we have $D_\top = \{x \mapsto \top \mid x \in \mathit{Vars}\}$ .

## Example:



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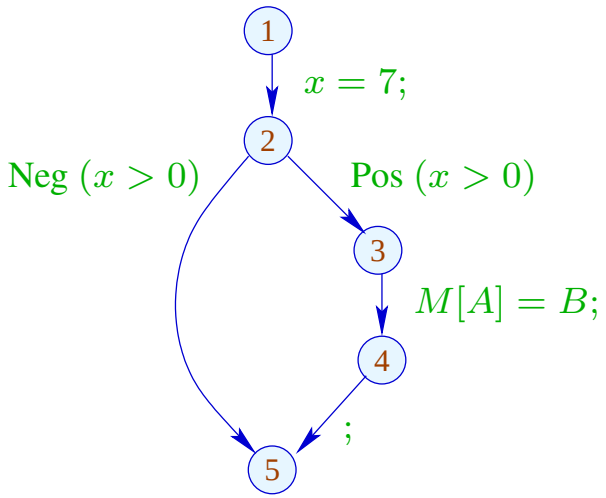


| | |
|---|---|
| 1 | $\{x \mapsto \top\}$ |
| 2 | $\{x \mapsto 7\}$ |
| 3 | $\{x \mapsto 7\}$ |
| 4 | $\{x \mapsto 7\}$ |
| 5 | $\bot \sqcup \{x \mapsto 7\} = \{x \mapsto 7\}$ |

The abstract effects of edges $[\![k]\!]^\sharp$ are again composed to form the effects of paths $\pi = k_1 \ldots k_r$ by:

$$[\![\pi]\!]^\sharp = [\![k_r]\!]^\sharp \circ \ldots \circ [\![k_1]\!]^\sharp \quad : \mathbb{D} \to \mathbb{D}$$

## Idea for Correctness: Abstract Interpretation

Cousot, Cousot 1977

Establish a description relation $\Delta$ between the concrete values and their descriptions with:

$$x \,\Delta\, a_1 \quad \wedge \quad a_1 \sqsubseteq a_2 \quad \Longrightarrow \quad x \,\Delta\, a_2$$

Concretization: $\gamma\, a = \{x \mid x \,\Delta\, a\}$

// returns the set of described values

(1) Values: $\Delta \subseteq \mathbb{Z} \times \mathbb{Z}^\top$

$$z \,\Delta\, a \quad \text{iff} \quad z = a \vee a = \top$$

Concretization:

$$\gamma\, a = \begin{cases} \{a\} & \text{if} \quad a \sqsubset \top \\ \mathbb{Z} & \text{if} \quad a = \top \end{cases}$$

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$$\gamma\, a = \begin{cases} \{a\} & \text{if} \quad a \sqsubset \top \\ \mathbb{Z} & \text{if} \quad a = \top \end{cases}$$

(2) Variable Bindings: $\Delta \subseteq (\mathit{Vars} \to \mathbb{Z}) \times (\mathit{Vars} \to \mathbb{Z}^\top)_\bot$

$$\rho \,\Delta\, D \quad \text{iff} \quad D \neq \bot \;\wedge\; \rho\, x \sqsubseteq D\, x \quad (x \in \mathit{Vars})$$

Concretization:

$$\gamma\, D = \begin{cases} \emptyset & \text{if} \quad D = \bot \\ \{\rho \mid \forall\, x :\ (\rho\, x)\ \Delta\ (D\, x)\} & \text{otherwise} \end{cases}$$

Example: $\{x \mapsto 1, y \mapsto -7\} \;\Delta\; \{x \mapsto \top, y \mapsto -7\}$

(3) States:

$$\Delta \;\subseteq\; ((\mathit{Vars} \to \mathbb{Z}) \times (\mathbb{N} \to \mathbb{Z})) \times (\mathit{Vars} \to \mathbb{Z}^{\top})_{\bot}$$

$$(\rho, \mu)\ \Delta\ D \qquad \text{iff} \qquad \rho\ \Delta\ D$$

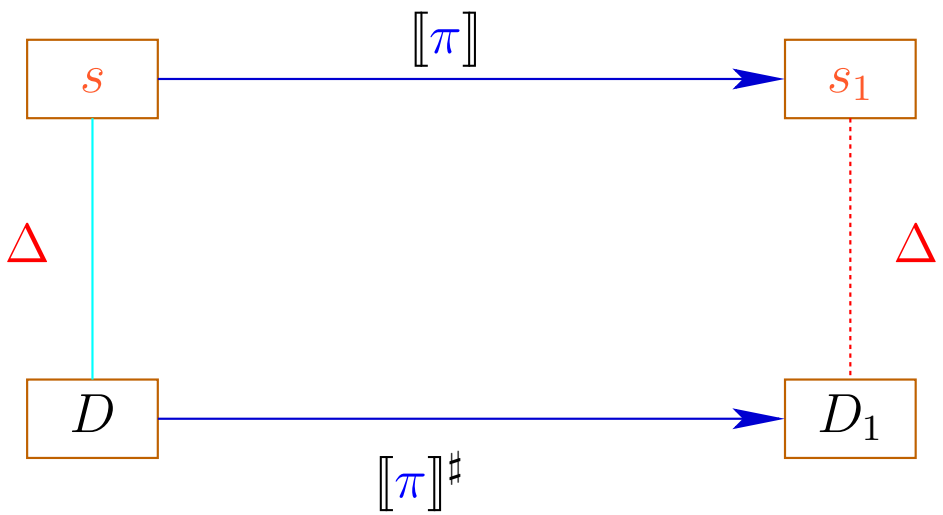
Concretization:

$$\gamma\, D = \begin{cases} \emptyset & \text{if} \quad D = \bot \\ \{(\rho, \mu) \mid \forall\, x :\ (\rho\, x)\ \Delta\ (D\, x)\} & \text{otherwise} \end{cases}$$

## We show correctness:

$(*)$ If $s \;\Delta\; D$ and $[\![\pi]\!]\, s$ is defined, then:

$$([\![\pi]\!]\, s) \;\Delta\; ([\![\pi]\!]^\sharp\, D)$$



The abstract semantics simulates the concrete semantics

In particular:

$$[\![\pi]\!]\, s \in \gamma\,([\![\pi]\!]^\sharp\, D)$$

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In practice, this means for example that $D\, x = -7$ implies:

$$\begin{array}{rrcll} & \rho'\, x & = & -7 & \text{for all} \quad \rho' \in \gamma\, D \\ \Longrightarrow & \rho_1\, x & = & -7 & \text{for} \quad (\rho_1, \_) = [\![\pi]\!]\, s \end{array}$$

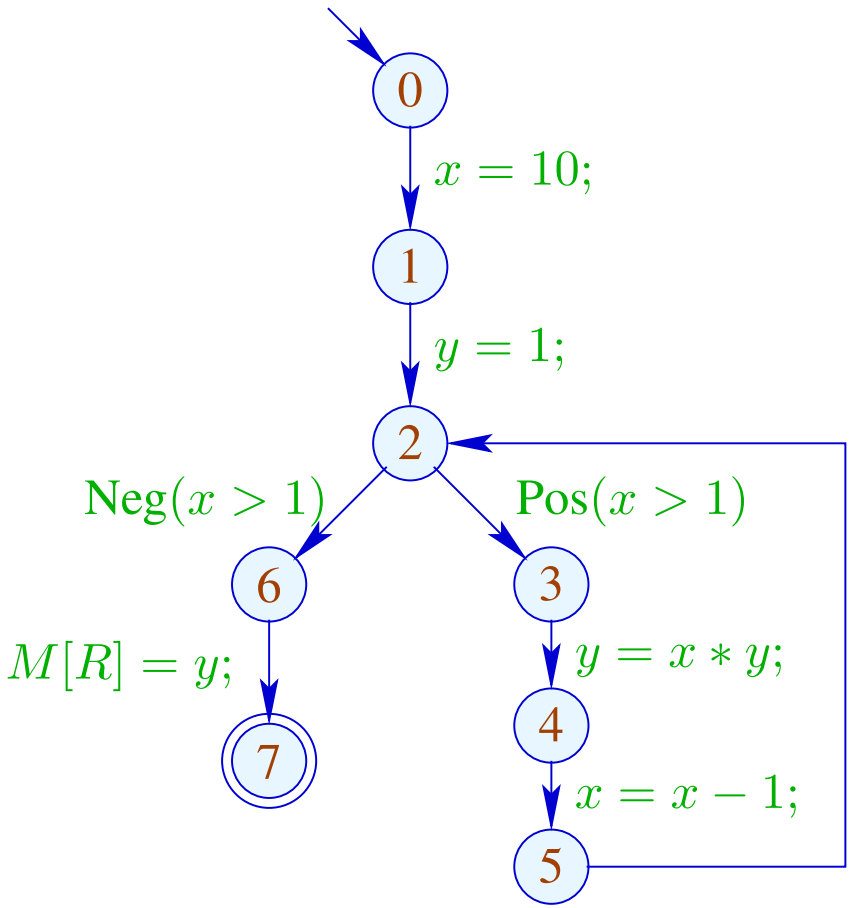
The MOP-Solution:

$$\mathcal{D}^*[v] \;=\; \bigsqcup\{[\![\pi]\!]^\sharp\; D_\top \mid \pi : start \to^* v\}$$

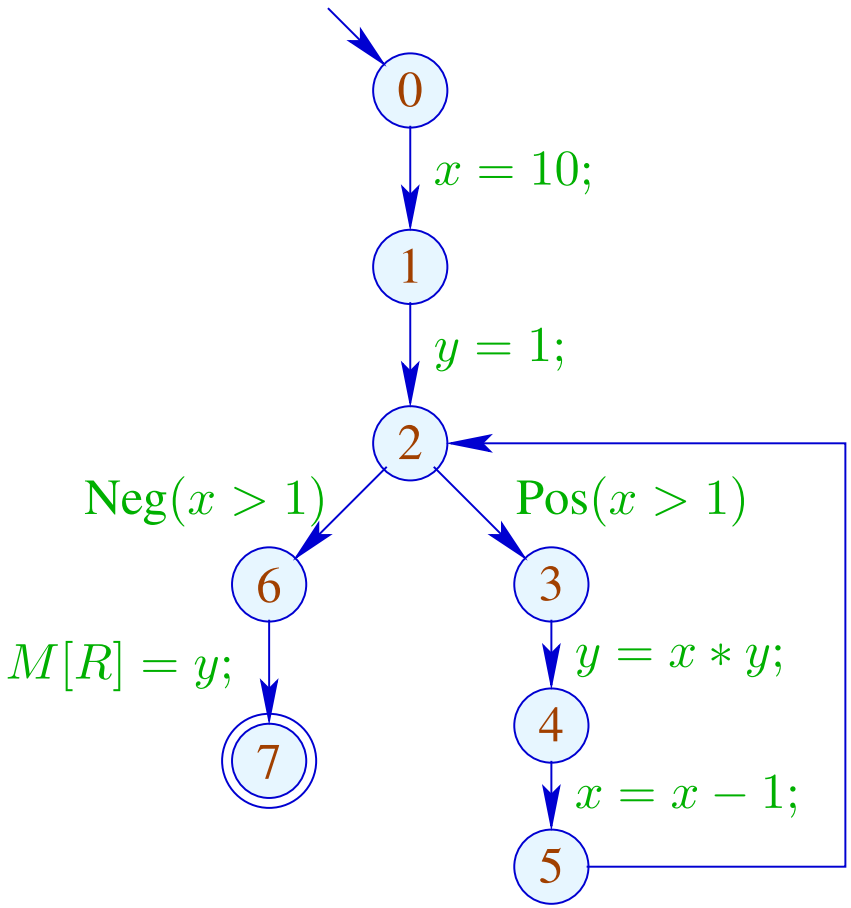
where $\quad D_\top\, x = \top \qquad (x \in Vars)\,.$

In order to approximate the MOP, we use our constraint system

# Example:

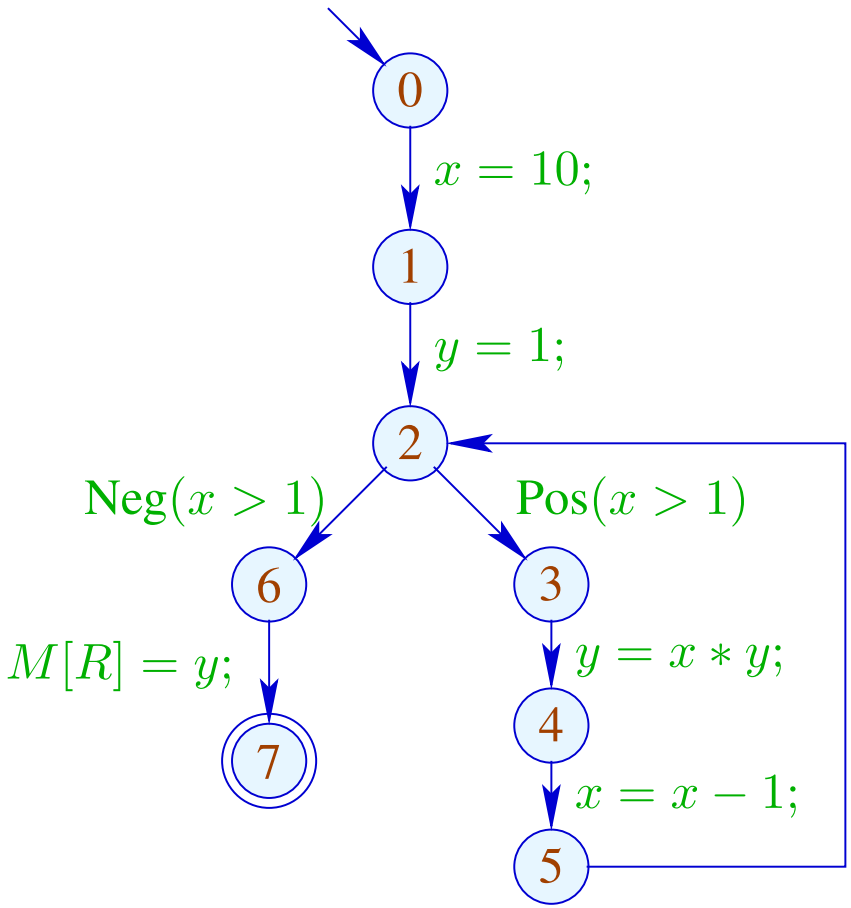


# Example:




| | 1 | |
|---|---|---|
| | $x$ | $y$ |
| 0 | $\top$ | $\top$ |
| 1 | 10 | $\top$ |
| 2 | 10 | 1 |
| 3 | 10 | 1 |
| 4 | 10 | 10 |
| 5 | 9 | 10 |
| 6 | $\bot$ | |
| 7 | $\bot$ | |

# Example:




| | 1 | | 2 | |
|---|---|---|---|---|
| | $x$ | $y$ | $x$ | $y$ |
| 0 | $\top$ | $\top$ | $\top$ | $\top$ |
| 1 | 10 | $\top$ | 10 | $\top$ |
| 2 | 10 | 1 | $\top$ | $\top$ |
| 3 | 10 | 1 | $\top$ | $\top$ |
| 4 | 10 | 10 | $\top$ | $\top$ |
| 5 | 9 | 10 | $\top$ | $\top$ |
| 6 | $\bot$ | | $\top$ | $\top$ |
| 7 | $\bot$ | | $\top$ | $\top$ |

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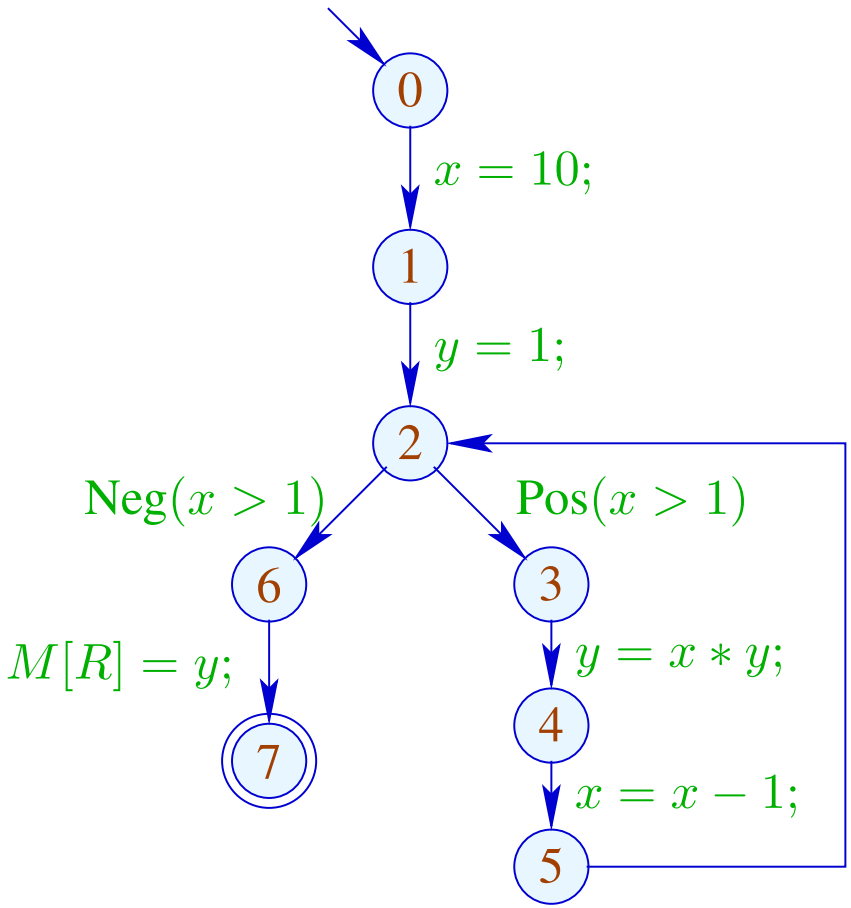


| | 1 | | 2 | | 3 | |
|---|---|---|---|---|---|---|
| | $x$ | $y$ | $x$ | $y$ | $x$ | $y$ |
| 0 | $\top$ | $\top$ | $\top$ | $\top$ | | |
| 1 | 10 | $\top$ | 10 | $\top$ | | |
| 2 | 10 | 1 | $\top$ | $\top$ | | |
| 3 | 10 | 1 | $\top$ | $\top$ | | |
| 4 | 10 | 10 | $\top$ | $\top$ | dito | |
| 5 | 9 | 10 | $\top$ | $\top$ | | |
| 6 | $\bot$ | | $\top$ | $\top$ | | |
| 7 | $\bot$ | | $\top$ | $\top$ | | |

## Concrete vs. Abstract Execution:

Although program and all initial values are given, abstract execution does not compute the result!

On the other hand, fixed-point iteration is guaranteed to terminate:

For $n$ program points and $m$ variables, we maximally need:
$n \cdot (m + 1)$ rounds

**Observation:** The effects of edges are not distributive!

Counterexample: $f = [\![x = x + y;]\!]^\sharp$

$$
\begin{array}{lrcl}
\text{Let} & D_1 & = & \{x \mapsto 2, y \mapsto 3\} \\
& D_2 & = & \{x \mapsto 3, y \mapsto 2\} \\
\\
\text{Then} & f\, D_1 \sqcup f\, D_2 & = & \{x \mapsto 5, y \mapsto 3\} \sqcup \{x \mapsto 5, y \mapsto 2\} \\
& & = & \{x \mapsto 5, y \mapsto \top\} \\
& & \neq & \{x \mapsto \top, y \mapsto \top\} \\
& & = & f\, \{x \mapsto \top, y \mapsto \top\} \\
& & = & f\, (D_1 \sqcup D_2)
\end{array}
$$

## We conclude:

The least solution $\mathcal{D}$ of the constraint system in general yields only an upper approximation of the MOP, i.e.,

$$\mathcal{D}^*[v] \quad \sqsubseteq \quad \mathcal{D}[v]$$

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The least solution $\mathcal{D}$ of the constraint system in general yields only an upper approximation of the MOP, i.e.,

$$\mathcal{D}^*[v] \sqsubseteq \mathcal{D}[v]$$

As an upper approximation, $\mathcal{D}[v]$ nonetheless describes the result of every program execution $\pi$ that reaches $v$ :

$$(\llbracket \pi \rrbracket (\rho, \mu)) \ \Delta \ (\mathcal{D}[v])$$

whenever $\llbracket \pi \rrbracket (\rho, \mu)$ is defined

# 6 Removing superfluous computations

A computation may be superfluous because

- the result is already available, $\longrightarrow$ available-expression analysis, or
- the result is not needed $\longrightarrow$ live-variable analysis.

## 6.1 Redundant computations

Idea:

If an expression at a program point is guaranteed to be computed to the value it had before, then

→ store this value after the first computation;

→ replace every further computation through a look-up

Question to be answered by static analysis: Is an expression available?

**Problem:** Identify sources of redundant computations!

**Example:**

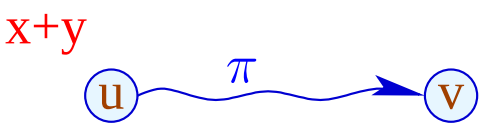
$$
\begin{array}{lll}
 & z = 1; \\
 & y = M[17]; \\
A: & x_1 = \boxed{y+z}; \\
 & \cdots \\
B: & x_2 = \boxed{y+z};
\end{array}
$$

$B$ is a redundant computation of the value of $\boxed{y+z}$, if

(1) $A$ is always executed before $B$; and

(2) $y$ and $z$ at $B$ have the same values as at $A$

Situation: The value of $x + y$ is computed at program point $u$




and a computation along path $\pi$ reaches $v$ where it evaluates again $x + y$

....

If $x$ and $y$ have not been modified in $\pi$, then evaluation of $x + y$ at $v$ returns the same value as evaluation at $u$.

This property can be checked at every edge in $\pi$.

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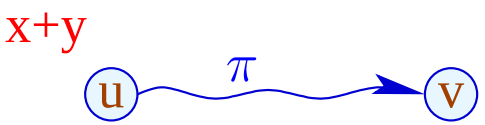


and a computation along path $\pi$ reaches $v$ where it evaluates again $x + y$ .... If $x$ and $y$ have not been modified in $\pi$, then evaluation of $x + y$ at $v$ is known to return the same value as evaluation at $u$

This property can be checked at every edge in $\pi$.

More efficient: Do this check for all expressions occurring in the program in parallel.

Assume that the expressions $A = \{e_1, \ldots, e_r\}$ are available at $u$.

.

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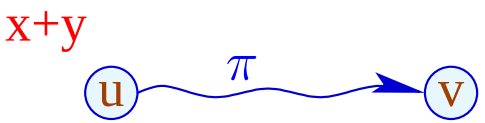


and a computation along path $\pi$ reaches $v$ where it evaluates again $x + y$ .... If $x$ and $y$ have not been modified in $\pi$, then evaluation of $x + y$ at $v$ must return the same value as evaluation at $u$.

This property can be checked at every edge in $\pi$.

**More efficient:** Do this check for all expressions occurring in the program in parallel.

Assume that the expressions $A = \{e_1, \ldots, e_r\}$ are available at $u$.

Every edge $k$ transforms this set into a set $[\![k]\!]^\sharp\, A$ of expressions whose values are available after execution of $k$.

$[\![k]\!]^\sharp\, A$ is the (abstract) edge effect associated with $k$

These edge effects can be composed to the effect of a path $\pi = k_1 \ldots k_r$:

$$[\![\pi]\!]^\sharp = [\![k_r]\!]^\sharp \circ \ldots \circ [\![k_1]\!]^\sharp$$

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The effect $[\![k]\!]^\sharp$ of an edge $k = (u, lab, v)$ only depends on the label $lab$, i.e., $[\![k]\!]^\sharp = [\![lab]\!]^\sharp$

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The effect $[\![k]\!]^\sharp$ of an edge $k = (u, lab, v)$ only depends on the label $lab$, i.e., $[\![k]\!]^\sharp = [\![lab]\!]^\sharp$ where:

$$\begin{array}{lcll} [\![;]\!]^\sharp\, A & = & A & \\ [\![true(e)]\!]^\sharp\, A & = & [\![false(e)]\!]^\sharp\, A = A \cup \{e\} & \\ [\![x = e;]\!]^\sharp\, A & = & (A \cup \{e\}) \backslash Expr_x & \text{where} \\ & & & Expr_x \text{ all expressions that contain } x \end{array}$$

$$\begin{array}{lcl} [\![x = M[e];]\!]^\sharp\, A & = & (A \cup \{e\}) \backslash Expr_x \\ [\![M[e_1] = e_2;]\!]^\sharp\, A & = & A \cup \{e_1, e_2\} \end{array}$$

$\rightarrow$ An expression is available at $v$ if it is available along all paths $\pi$ to $v$.

$\rightarrow$ For every such path $\pi$, the analysis determines the set of expressions that are available along $\pi$.

$\rightarrow$ Initially at program start, nothing is available.

$\rightarrow$ The analysis computes the intersection of the availability sets as safe information.

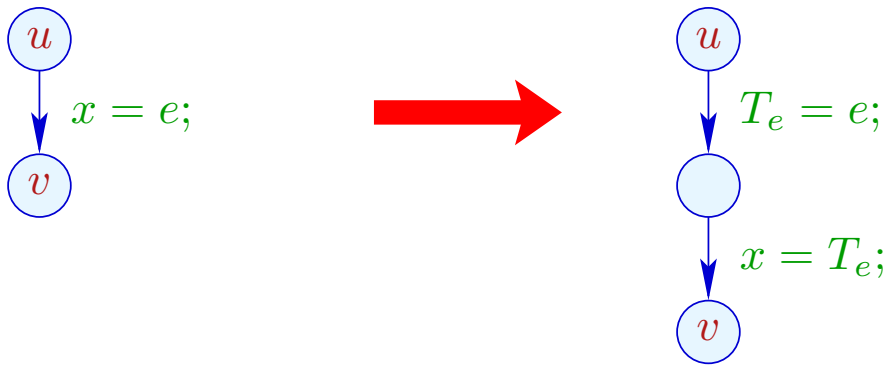
$\Longrightarrow$ For each node $v$, we need the set:

$$\mathcal{A}[v] \quad = \quad \bigcap \{ [\![\pi]\!]^\sharp \emptyset \mid \pi : start \rightarrow^* v \}$$

## How does a compiler exploit this information?

## Transformation UT (unique temporaries):

We provide a novel register $T_e$ as storage for the values of $e$:



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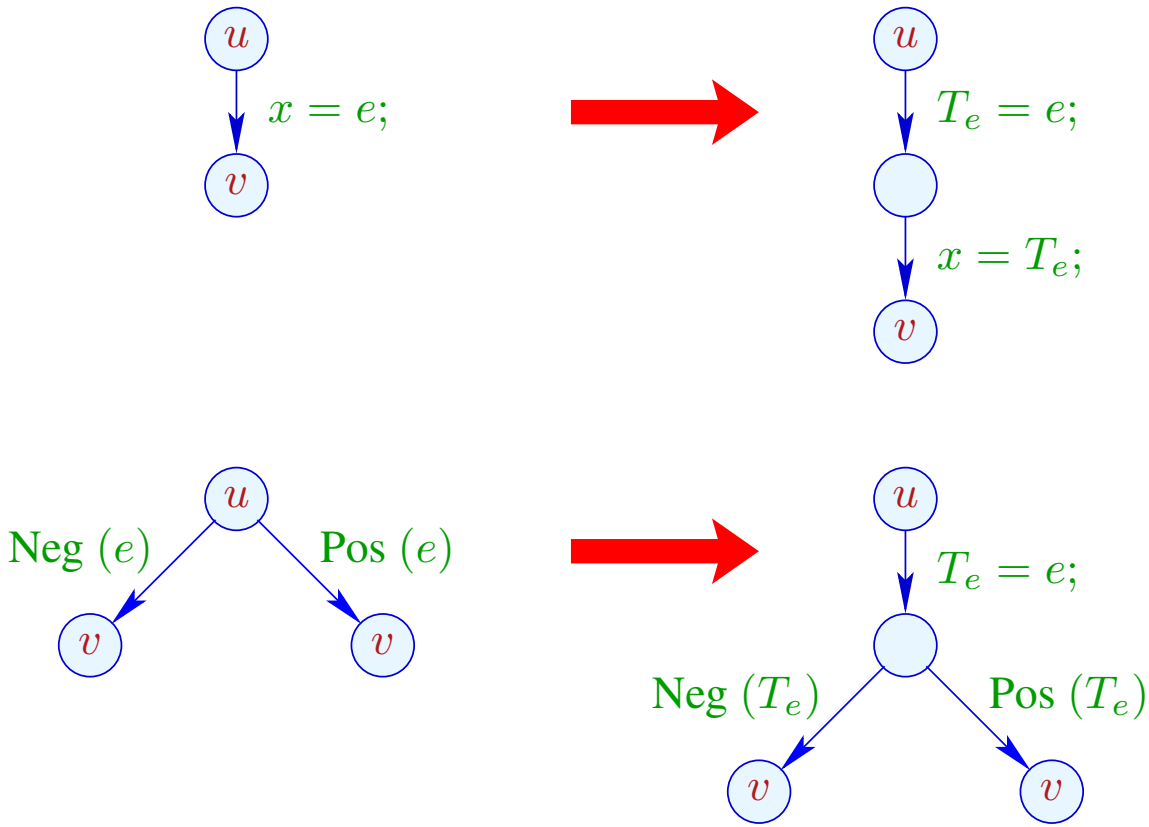
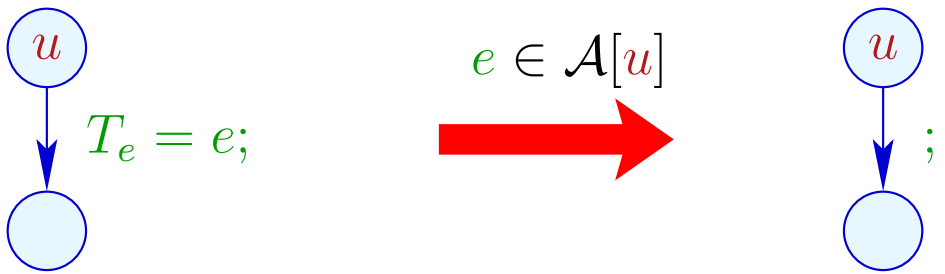



... analogously for $R = M[e];$ and $M[e_1] = e_2;$.

## Transformation AEE (available expression elimination):

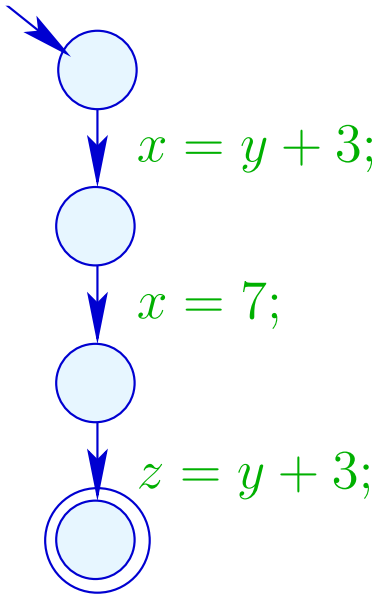
If $e$ is available at program point $u$, then $e$ need not be re-evaluated:




We replace the assignment with $Nop$.

## Example:

$$
\begin{aligned}
x &= y + 3; \\
x &= 7; \\
z &= y + 3;
\end{aligned}
$$



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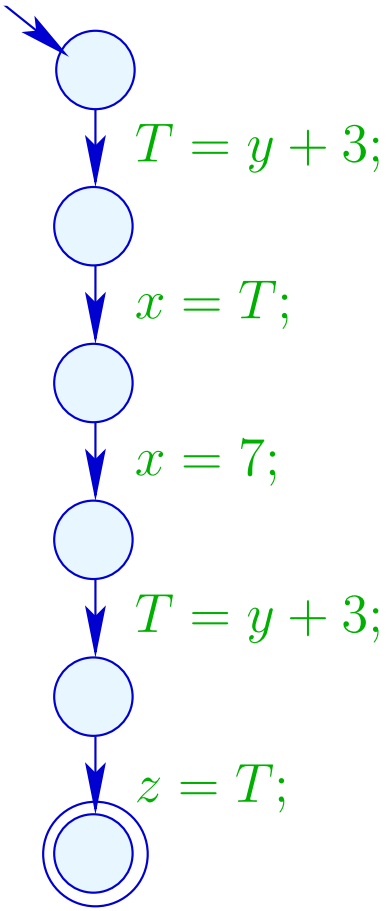

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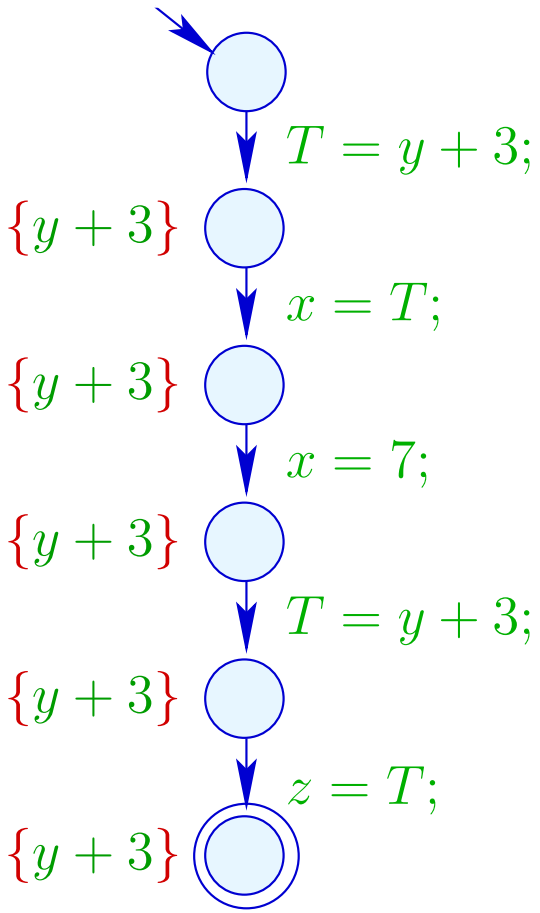

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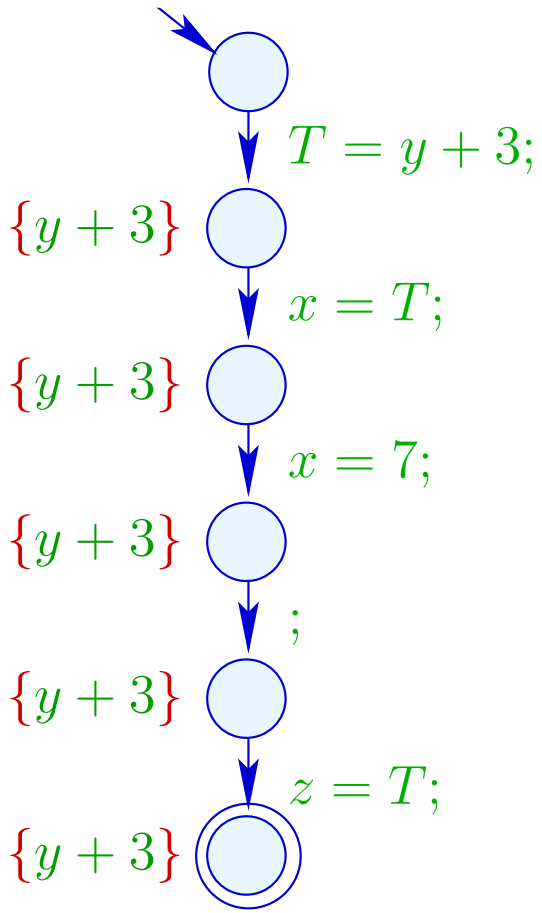

## Warning:

Transformation UT is only meaningful for assignments $x = e;$ where:

- $\rightarrow$ $x \notin Vars(e)$; why?
- $\rightarrow$ $e \notin Vars$; why?
- $\rightarrow$ the evaluation of $e$ is non-trivial; why?

## Warning:

Transformation UT is only meaningful for assignments $x = e;$ where:

- $\rightarrow$ $x \notin Vars(e)$; otherwise $e$ is not available afterwards.
- $\rightarrow$ $e \notin Vars$; otherwise values are shuffled around
- $\rightarrow$ the evaluation of $e$ is non-trivial; otherwise the efficiency of the code is decreased.

Open question ...

## Question:

How do we compute $\mathcal{A}[u]$ for every program point $u$?

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How can we compute $\mathcal{A}[u]$ for every program point? $u$

We collect all constraints on the values of $\mathcal{A}[u]$ into a system of constraints:

$$\begin{array}{lcll} \mathcal{A}[start] & \subseteq & \emptyset & \\ \mathcal{A}[v] & \subseteq & [\![k]\!]^\sharp\,(\mathcal{A}[u]) & k = (u, \_, v) \quad \text{edge} \end{array}$$

Why $\subseteq$?

## Question:

How can we compute $\mathcal{A}[u]$ for every program point? $u$

## Idea:

We collect all constraints on the values of $\mathcal{A}[u]$ into a system of constraints:

$$\begin{array}{lcll} \mathcal{A}[start] & \subseteq & \emptyset & \\ \mathcal{A}[v] & \subseteq & [\![k]\!]^\sharp\,(\mathcal{A}[u]) & k = (u, \_, v) \quad \text{edge} \end{array}$$

Why $\subseteq$?

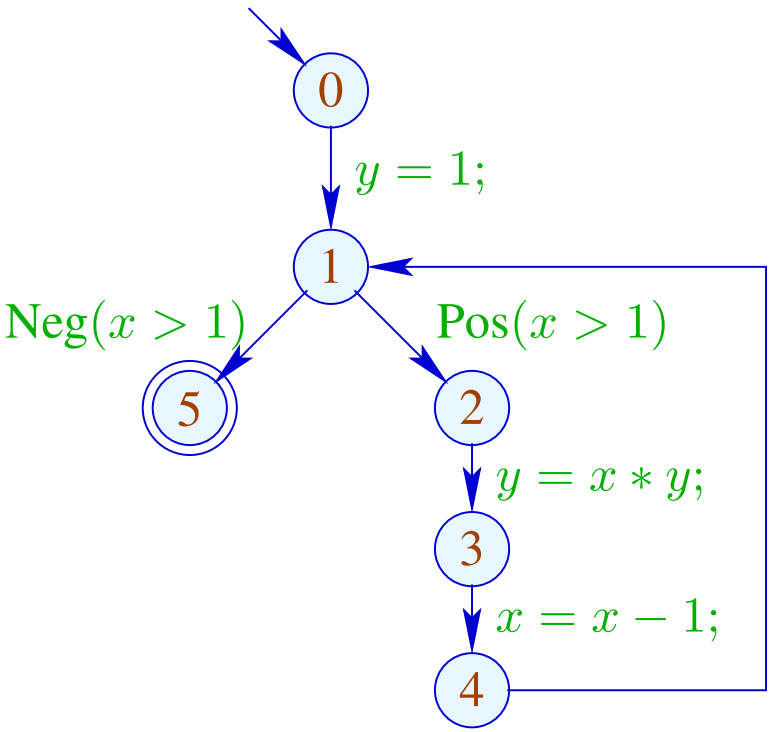
Then combine all constraints for each variable $v$ by applying the least-upper-bound operator $\longrightarrow$

$$\mathcal{A}[v] \quad \subseteq \quad \bigcap\{[\![k]\!]^\sharp\,(\mathcal{A}[u]) \mid k = (u, \_, v) \;\; \text{edge}\}$$

## Wanted:

- a greatest solution (why greatest?)
- an algorithm that computes this solution

## Example:



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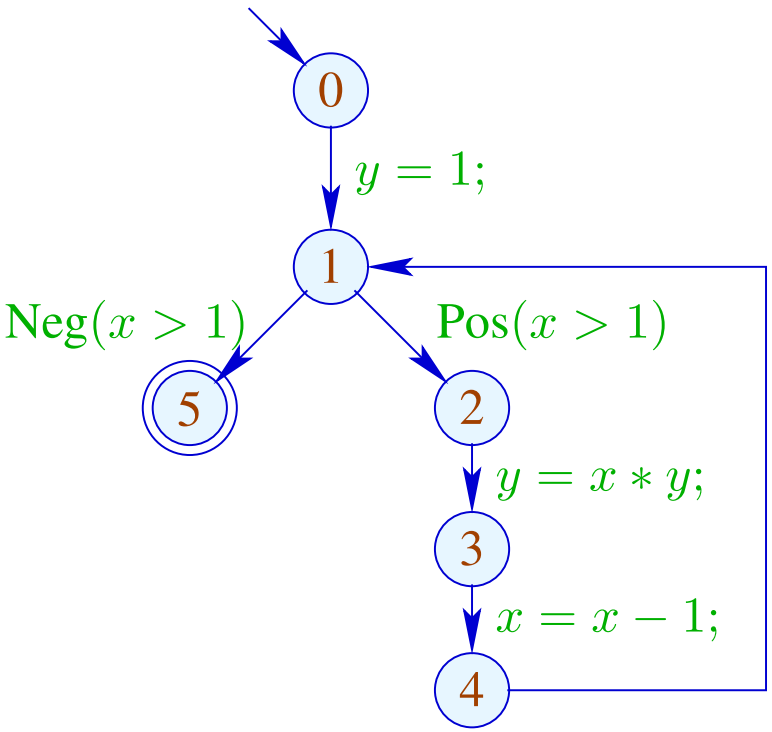


$$\mathcal{A}[0] \quad \subseteq \quad \emptyset$$

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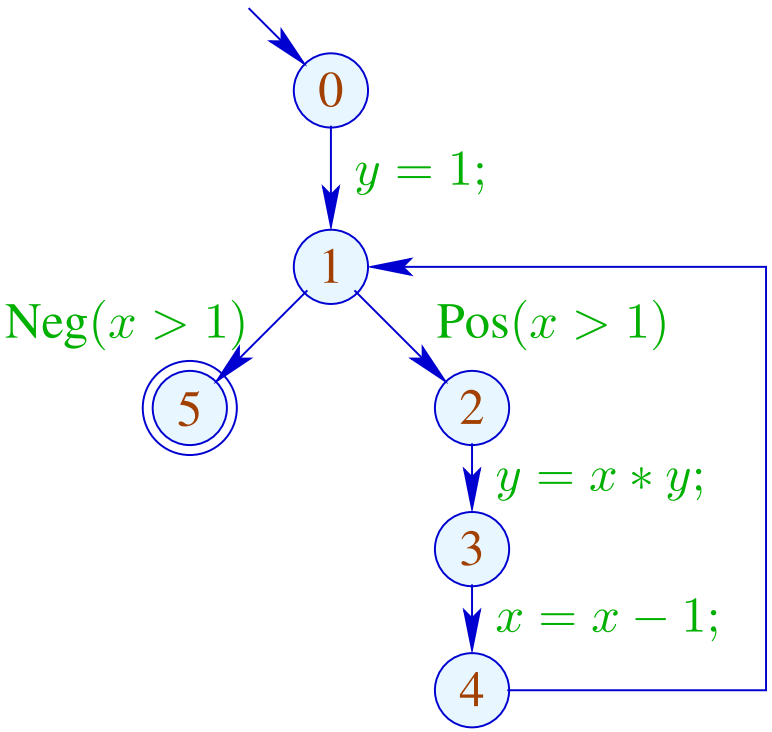


$$\begin{aligned} \mathcal{A}[0] &\subseteq \emptyset \\ \mathcal{A}[1] &\subseteq (\mathcal{A}[0] \cup \{1\}) \backslash Expr_y \\ \mathcal{A}[1] &\subseteq \mathcal{A}[4] \end{aligned}$$

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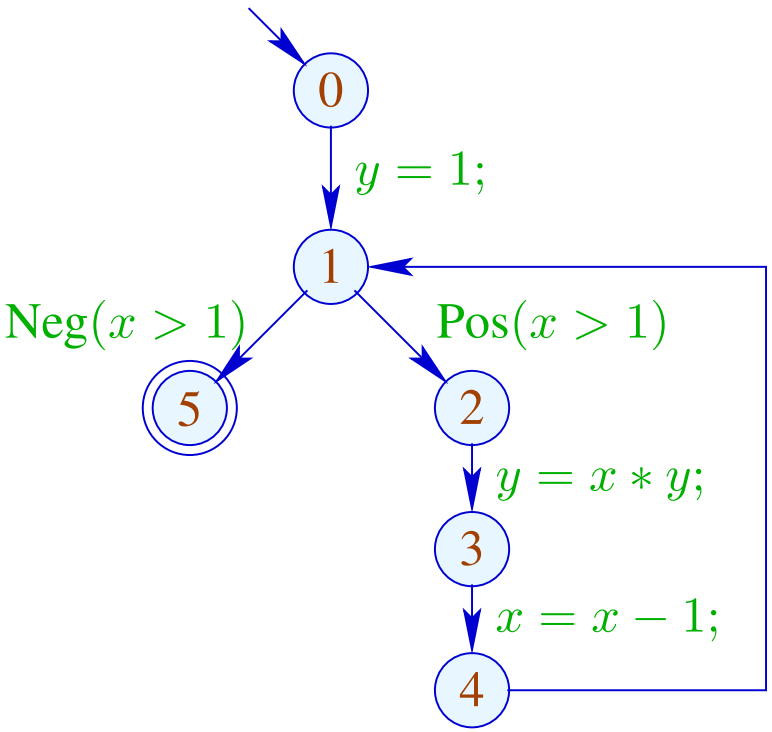
## Example:

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\end{aligned}
$$

## Wanted:

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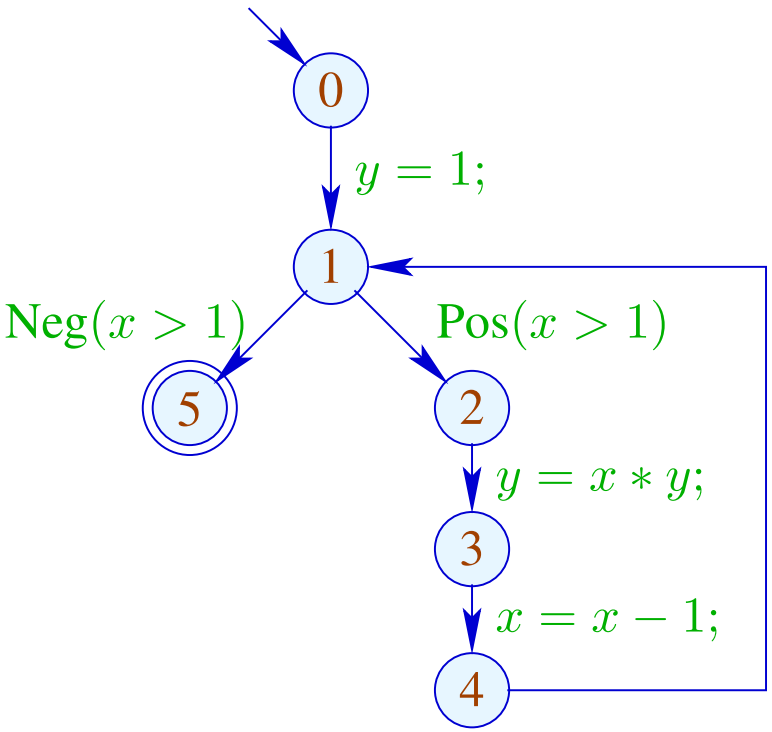
## Example:

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$$

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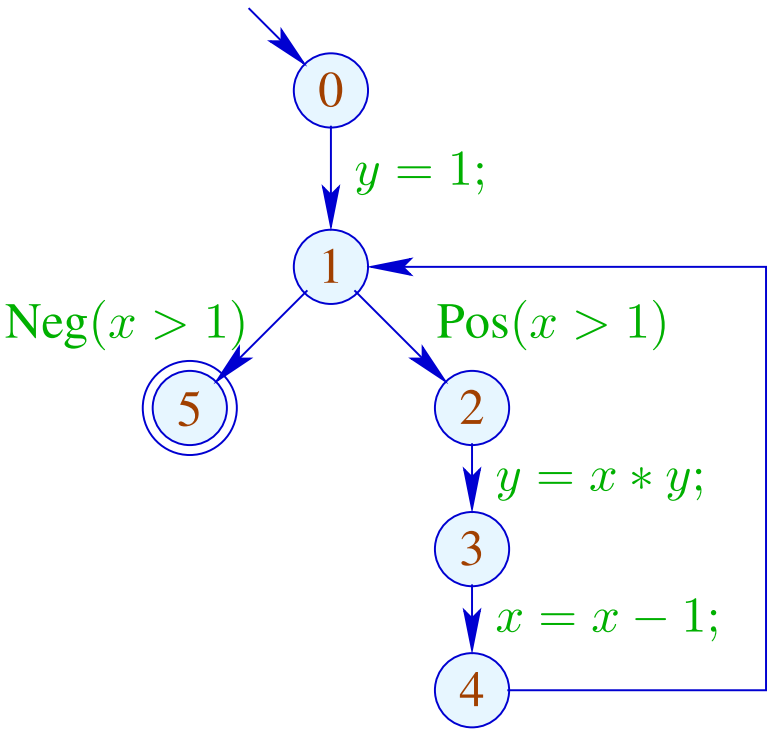


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\mathcal{A}[3] &\subseteq (\mathcal{A}[2] \cup \{x * y\}) \backslash Expr_y \\
\mathcal{A}[4] &\subseteq (\mathcal{A}[3] \cup \{x - 1\}) \backslash Expr_x
\end{aligned}
$$

## Wanted:

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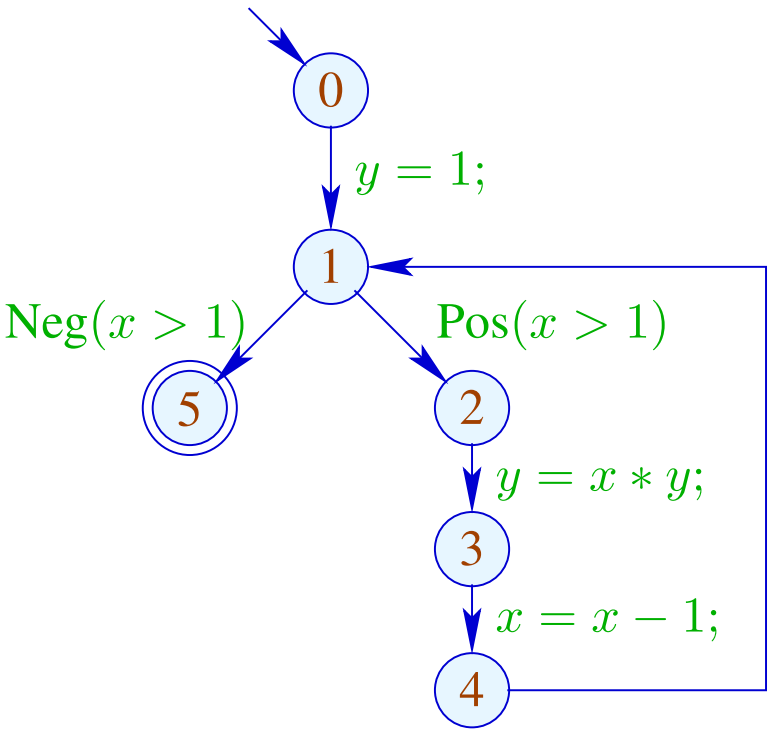
## Example:

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\begin{aligned}
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\mathcal{A}[1] &\subseteq \mathcal{A}[4] \\
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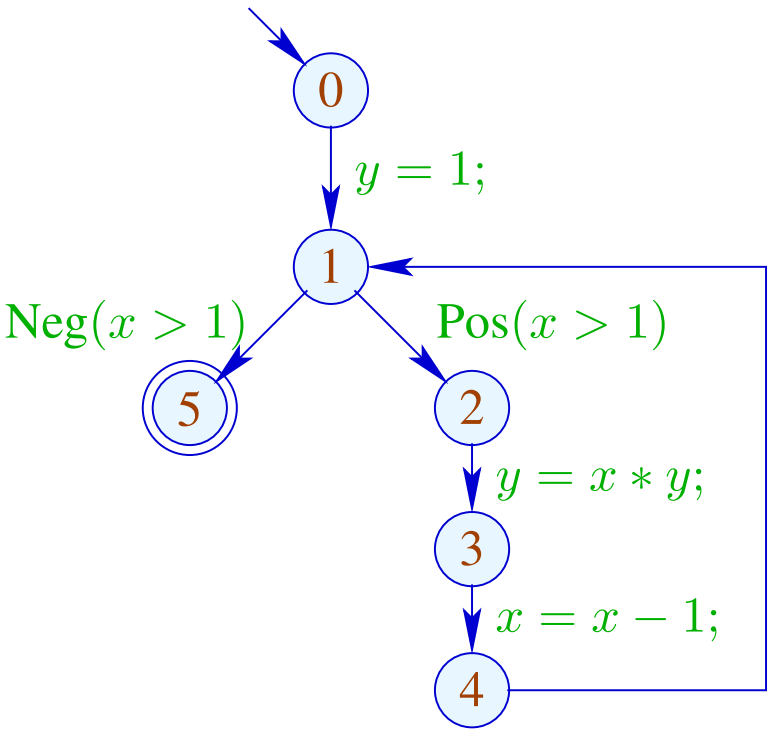
\mathcal{A}[3] &\subseteq (\mathcal{A}[2] \cup \{x * y\}) \backslash Expr_y \\
\mathcal{A}[4] &\subseteq (\mathcal{A}[3] \cup \{x - 1\}) \backslash Expr_x \\
\mathcal{A}[5] &\subseteq \mathcal{A}[1] \cup \{x > 1\}
\end{aligned}
$$

## Wanted:

- a greatest solution,
- an algorithm that computes this solution.

## Example:




## Solution:

$$
\begin{aligned}
\mathcal{A}[0] &= \emptyset \\
\mathcal{A}[1] &= \{1\} \\
\mathcal{A}[2] &= \{1, x > 1\} \\
\mathcal{A}[3] &= \{1, x > 1\} \\
\mathcal{A}[4] &= \{1\} \\
\mathcal{A}[5] &= \{1, x > 1\}
\end{aligned}
$$

## Observation:

- Again, the possible values for $\mathcal{A}[u]$ form a complete lattice:

$$\mathbb{D} = 2^{Expr} \quad \text{with} \quad B_1 \sqsubseteq B_2 \quad \text{iff} \quad B_1 \supseteq B_2$$

- The order on the lattice elements indicates what is better information,
  more available expressions may allow more optimizations

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- The order on the lattice elements indicates what is better information,
  more available expressions may allow more optimizations
- The functions $[\![k]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ have the form $f_i\, x = a_i \cap x \cup b_i$.
  They are called *gen/kill* functions — $\cap$ kills, $\cup$ generates.
- they are monotonic, i.e.,

$$[\![k]\!]^\sharp(B_1) \sqsubseteq [\![k]\!]^\sharp(B_2) \quad \text{iff} \quad B_1 \sqsubseteq B_2$$

The operations "$\circ$", "$\sqcup$" and "$\sqcap$" can be explicitly defined by:

$$
\begin{aligned}
(f_2 \circ f_1)\, x &= \boxed{a_1 \cap a_2} \cap x \cup \boxed{a_2 \cap b_1 \cup b_2} \\
(f_1 \sqcup f_2)\, x &= \boxed{(a_1 \cup a_2)} \cap x \cup \boxed{b_1 \cup b_2} \\
(f_1 \sqcap f_2)\, x &= \boxed{(a_1 \cup b_1) \cap (a_2 \cup b_2)} \cap x \cup \boxed{b_1 \cap b_2}
\end{aligned}
$$

## 6.2 Removing Assignments to Dead Variables

Example:

$$
\begin{aligned}
1: &\quad x = y + 2; \\
2: &\quad y = 5; \\
3: &\quad x = y + 3;
\end{aligned}
$$

The value of $x$ at program points $1, 2$ is overwritten before it can be used.

Therefore, we call the variable $x$ dead at these program points.

## Note:

- → Assignments to dead variables can be removed.
- → Such inefficiencies may originate from other transformations.

## Note:

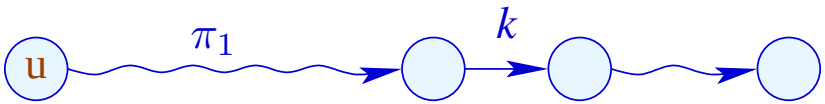
$\rightarrow$ Assignments to dead variables can be removed.

$\rightarrow$ Such inefficiencies may originate from other transformations.

## Formal Definition:

The variable $x$ is called live at $u$ along a path $\pi$ starting at $u$ if $\pi$ can be decomposed into $\pi = \pi_1 \, k \, \pi_2$ such that:

- $k$ is a use of $x$ and
- $\pi_1$ does not contain a definition of $x$.

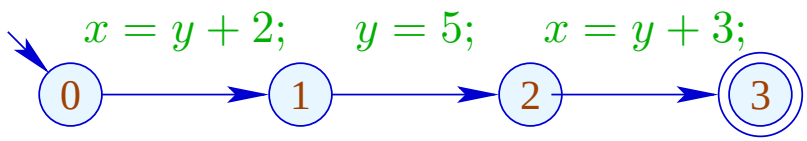



Thereby, the set of all defined or used variables at an edge $k = (\_, lab, \_)$ is defined by

| $lab$ | used | defined |
|---|---|---|
| ; | $\emptyset$ | $\emptyset$ |
| $\text{true}\,(e)$ | $Vars\,(e)$ | $\emptyset$ |
| $\text{false}\,(e)$ | $Vars\,(e)$ | $\emptyset$ |
| $x = e;$ | $Vars\,(e)$ | $\{x\}$ |
| $x = M[e];$ | $Vars\,(e)$ | $\{x\}$ |
| $M[e_1] = e_2;$ | $Vars\,(e_1) \cup Vars\,(e_2)$ | $\emptyset$ |

A variable $x$ which is not live at $u$ along $\pi$ is called dead at $u$ along $\pi$.

## Example:




Then we observe:

| | live | dead |
|---|---|---|
| 0 | $\{y\}$ | $\{x\}$ |
| 1 | $\emptyset$ | $\{x, y\}$ |
| 2 | $\{y\}$ | $\{x\}$ |
| 3 | $\emptyset$ | $\{x, y\}$ |

The variable $x$ is live at $u$ if $x$ is live at $u$ along some path to the exit . Otherwise, $x$ is called dead at $u$ .

The variable $x$ is live at $u$ if $x$ is live at $u$ along some path to the exit. Otherwise, $x$ is called dead at $u$.

## Question:

How can the sets of all dead/live variables be computed for every $u$?

The variable $x$ is live at $u$ if $x$ is live at $u$ along some path to the exit. Otherwise, $x$ is called dead at $u$ .

## Question:

How can the sets of all dead/live variables be computed for every $u$?

## Idea:

For every edge $k = (u, \_, v)$ , define a function $[\![k]\!]^\sharp$ which transforms the set of variables that are live at $v$ into the set of variables that are live at $u$.

Note: Edge transformers go "backwards"!

Let $\mathbb{L} = 2^{Vars}$ .

For $k = (\_, lab, \_)$ , define $[\![k]\!]^\sharp = [\![lab]\!]^\sharp$ by:

$$
\begin{array}{lcl}
[\![;]\!]^\sharp\, L & = & L \\
[\![\mathrm{true}(e)]\!]^\sharp\, L & = & [\![\mathrm{false}(e)]\!]^\sharp\, L \;=\; L \cup Vars(e) \\
[\![x = e;]\!]^\sharp\, L & = & (L \backslash \{x\}) \cup Vars(e) \\
[\![x = M[e];]\!]^\sharp\, L & = & (L \backslash \{x\}) \cup Vars(e) \\
[\![M[e_1] = e_2;]\!]^\sharp\, L & = & L \cup Vars(e_1) \cup Vars(e_2)
\end{array}
$$

Let $\mathbb{L} = 2^{Vars}$ .

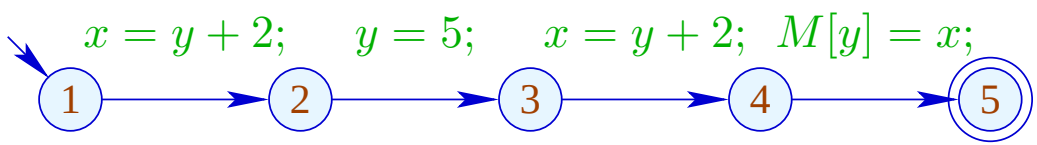
For $k = (\_, lab, \_)$ , define $[\![k]\!]^\sharp = [\![lab]\!]^\sharp$ by:

$$
\begin{array}{lcl}
[\![;]\!]^\sharp\, L & = & L \\
[\![\mathrm{true}(e)]\!]^\sharp\, L & = & [\![\mathrm{false}(e)]\!]^\sharp\, L \;=\; L \cup Vars(e) \\
[\![x = e;]\!]^\sharp\, L & = & (L \backslash \{x\}) \cup Vars(e) \\
[\![x = M[e];]\!]^\sharp\, L & = & (L \backslash \{x\}) \cup Vars(e) \\
[\![M[e_1] = e_2;]\!]^\sharp\, L & = & L \cup Vars(e_1) \cup Vars(e_2)
\end{array}
$$

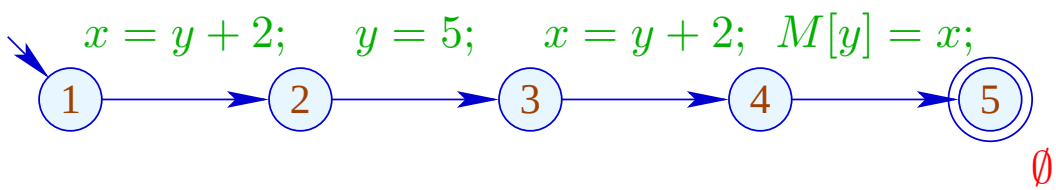
$[\![k]\!]^\sharp$ can again be composed to the effects of $[\![\pi]\!]^\sharp$ of paths $\pi = k_1 \ldots k_r$ by:

$$[\![\pi]\!]^\sharp = [\![k_1]\!]^\sharp \circ \ldots \circ [\![k_r]\!]^\sharp$$

We verify that these definitions are meaningful



We verify that these definitions are meaningful



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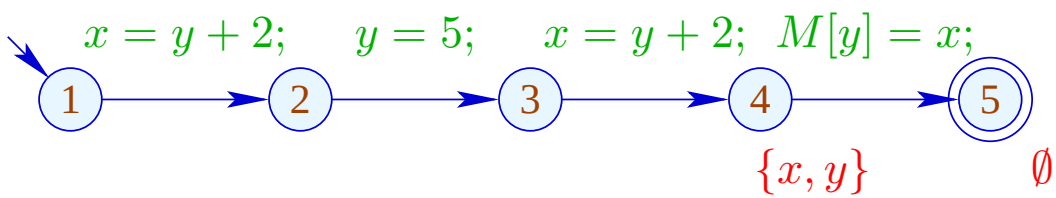

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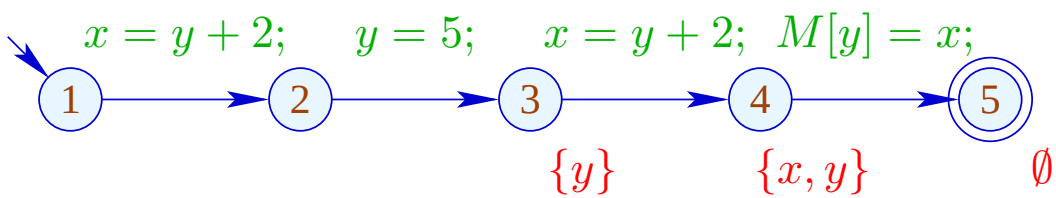

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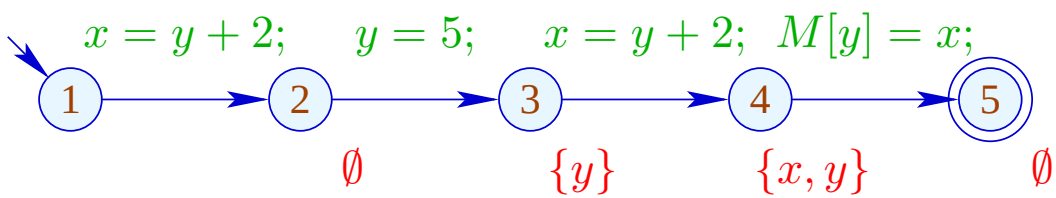

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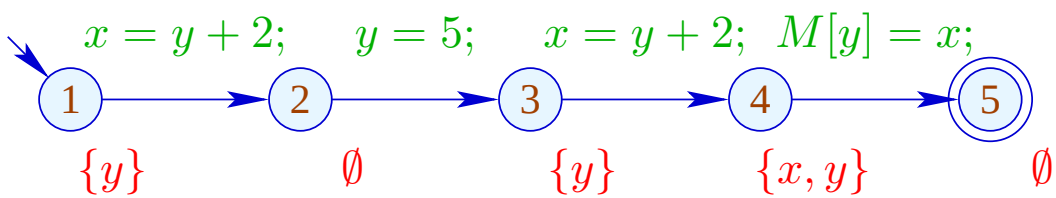


A variable is live at a program point $u$ if there is at least one path from $u$ to program exit on which it is live.

The set of variables which are live at $u$ therefore is given by:

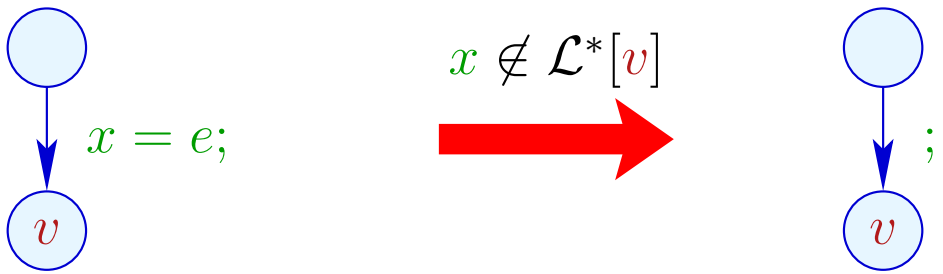
$$\mathcal{L}^*[u] \;=\; \bigcup\{[\![\pi]\!]^\sharp\emptyset \mid \pi : u \to^* stop\}$$

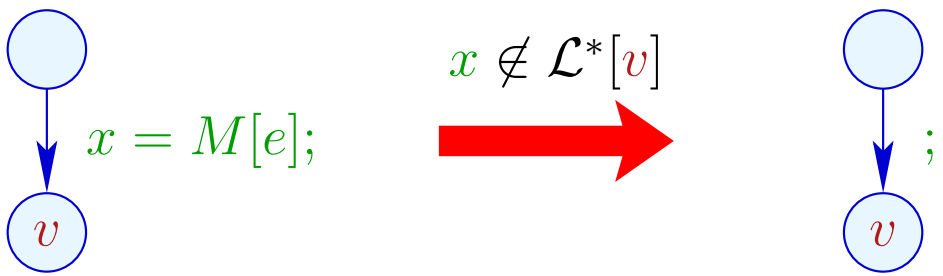
No variables are assumed to be live at program exit.

As partial order for $\mathbb{L}$ we use $\sqsubseteq \;=\; \subseteq$. why?

So, the least upper bound is $\bigcup$. why?

# Transformation DE (Dead assignment Elimination):



## Correctness Proof:

- → Correctness of the effects of edges: If $L$ is the set of variables which are live at the exit of the path $\pi$, then $[\![\pi]\!]^\sharp L$ is the set of variables which are live at the beginning of $\pi$.
- → Correctness of the transformation along a path: If the value of a variable is accessed, this variable is necessarily live. The value of dead variables thus is irrelevant.
- → Correctness of the transformation: In any execution of the transformed programs, the live variables always receive the same values as in the original program.

## Computation of the sets $\mathcal{L}^*[u]$ :

(1) Collecting constraints:

$$\begin{array}{lll} \mathcal{L}[stop] & \supseteq & \emptyset \\ \mathcal{L}[u] & \supseteq & [\![k]\!]^\sharp\,(\mathcal{L}[v]) \qquad k = (u, \_, v) \quad \text{edge} \end{array}$$

(2) Solving the constraint system by means of RR iteration.

Since $\mathbb{L}$ is finite, the iteration will terminate

(3) If the exit is (formally) reachable from every program point, then the least solution $\mathcal{L}$ of the constraint system equals $\mathcal{L}^*$ since all $[\![k]\!]^\sharp$ are distributive

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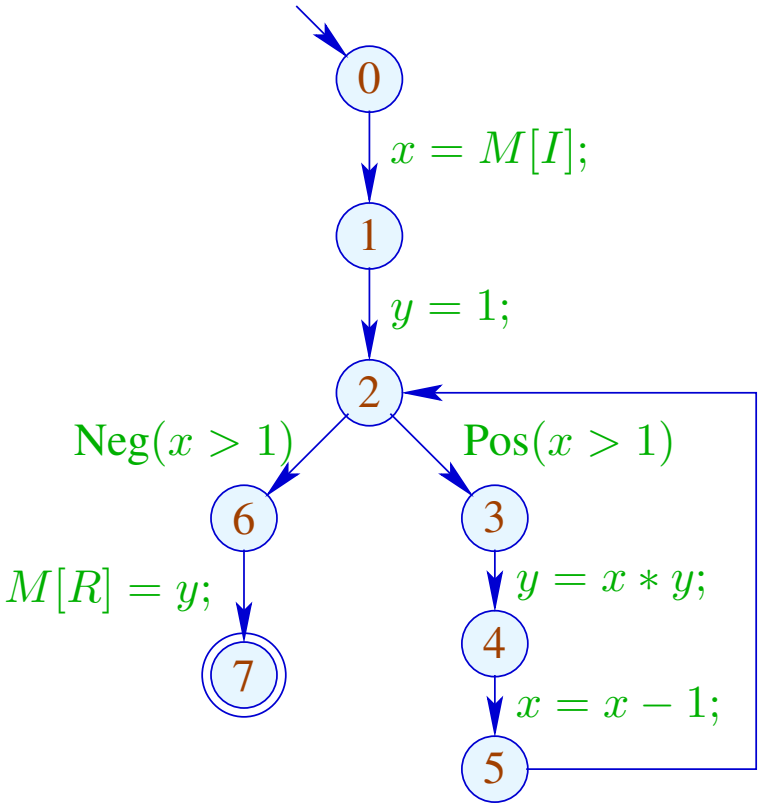
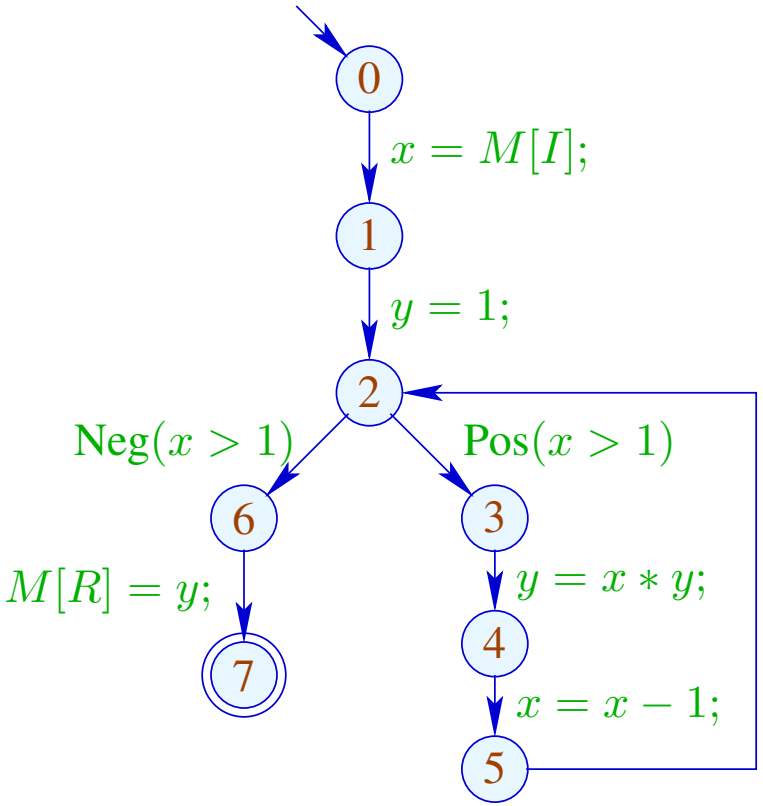
Note: The information is propagated backwards!

# Example:




$$
\begin{aligned}
\mathcal{L}[0] &\supseteq (\mathcal{L}[1] \backslash \{x\}) \cup \{I\} \\
\mathcal{L}[1] &\supseteq \mathcal{L}[2] \backslash \{y\} \\
\mathcal{L}[2] &\supseteq (\mathcal{L}[6] \cup \{x\}) \cup (\mathcal{L}[3] \cup \{x\}) \\
\mathcal{L}[3] &\supseteq (\mathcal{L}[4] \backslash \{y\}) \cup \{x, y\} \\
\mathcal{L}[4] &\supseteq (\mathcal{L}[5] \backslash \{x\}) \cup \{x\} \\
\mathcal{L}[5] &\supseteq \mathcal{L}[2] \\
\mathcal{L}[6] &\supseteq \mathcal{L}[7] \cup \{y, R\} \\
\mathcal{L}[7] &\supseteq \emptyset
\end{aligned}
$$

# Example:

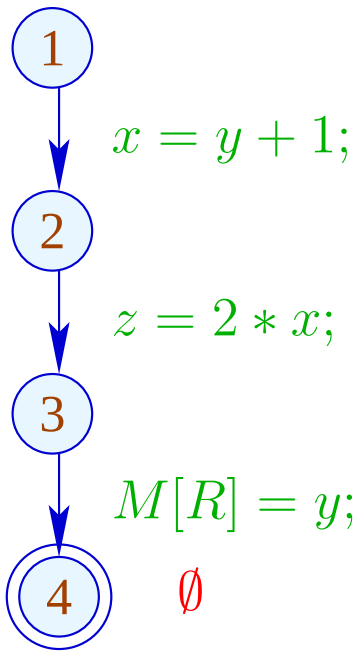



| | 1 | 2 |
|---|---|---|
| 7 | $\emptyset$ | |
| 6 | $\{y, R\}$ | |
| 2 | $\{x, y, R\}$ | dito |
| 5 | $\{x, y, R\}$ | |
| 4 | $\{x, y, R\}$ | |
| 3 | $\{x, y, R\}$ | |
| 1 | $\{x, R\}$ | |
| 0 | $\{I, R\}$ | |

The left-hand side of no assignment is dead

## Caveat:

Removal of assignments to dead variables may kill further variables:



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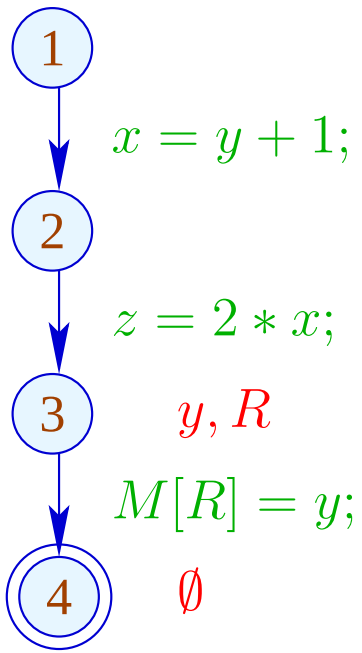

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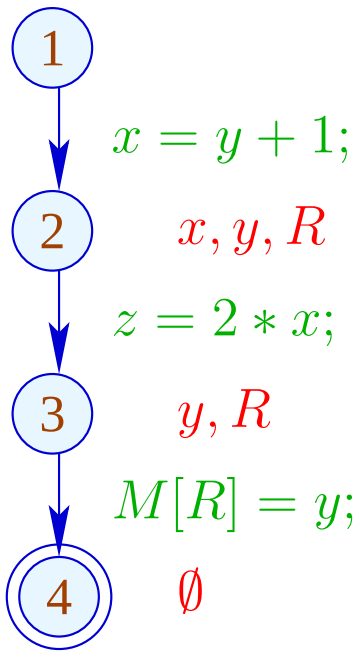

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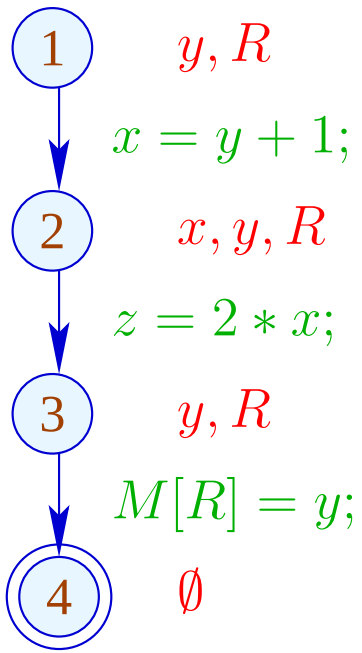

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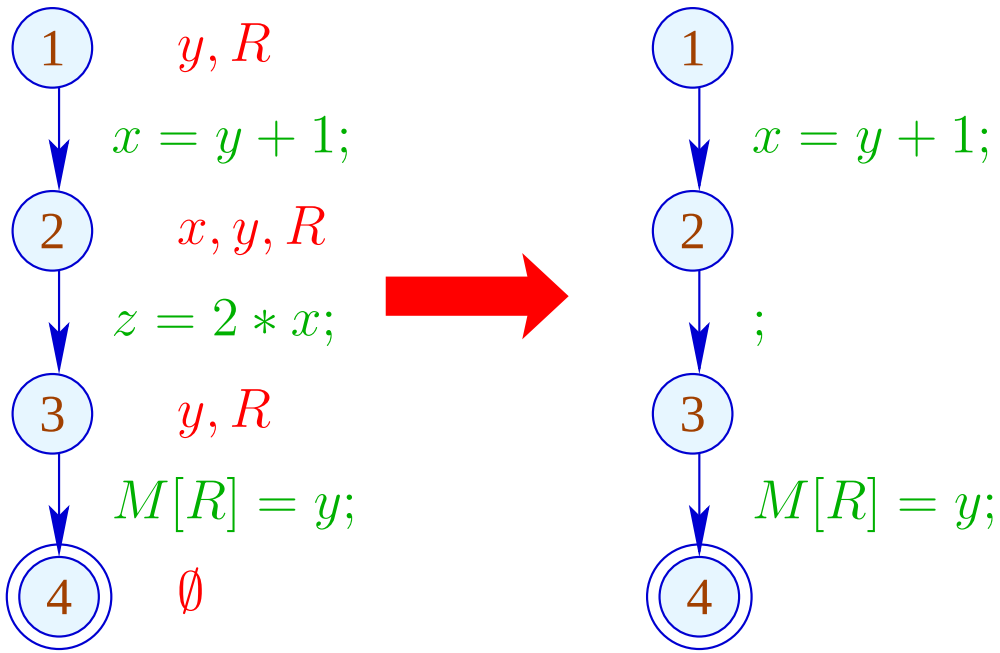

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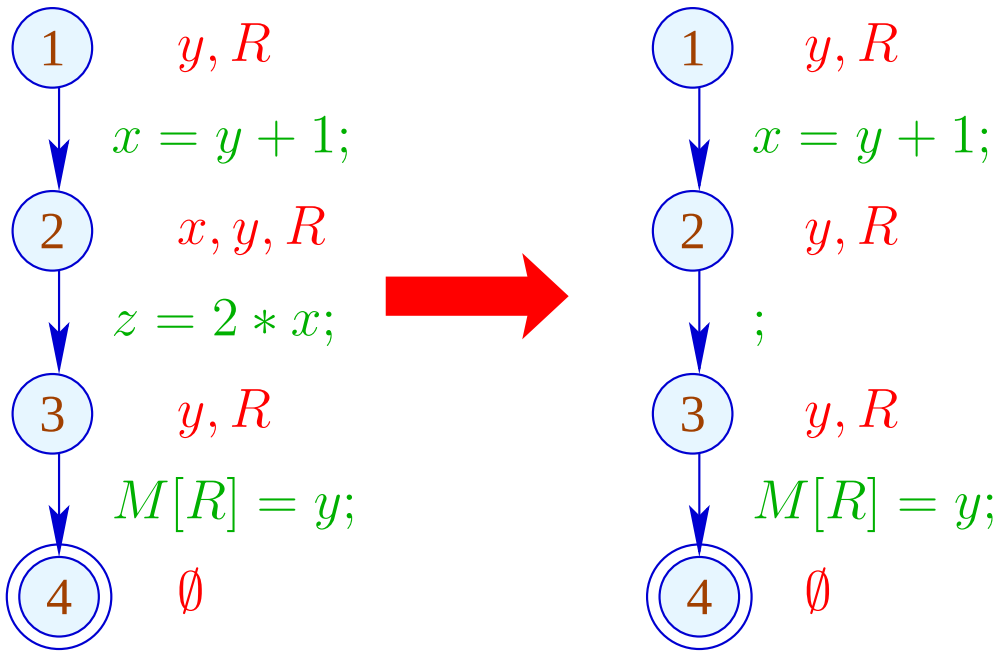

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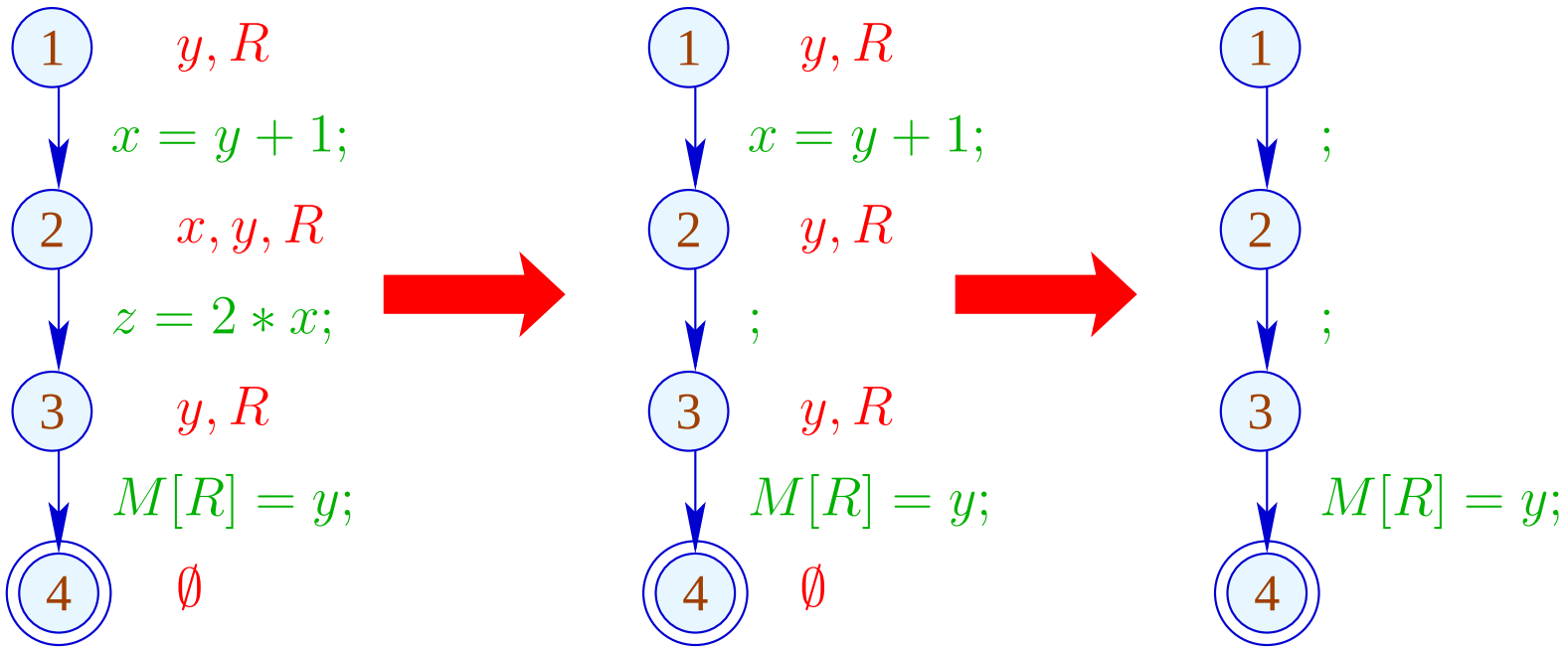


Re-analyzing the program is inconvenient

Idea: Analyze true liveness!

$x$ is called truly live at $u$ along a path $\pi$, either

if $\pi$ can be decomposed into $\pi = \pi_1 \, k \, \pi_2$ such that:

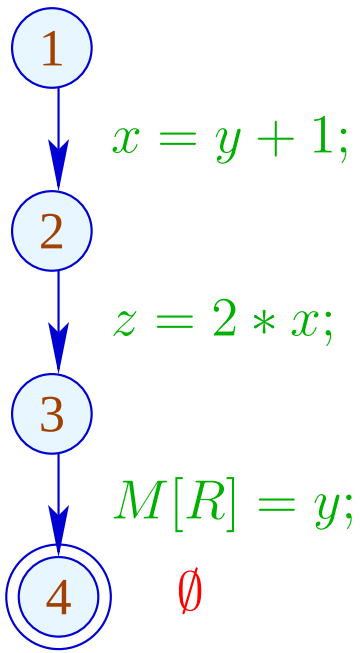
- $k$ is a true use of $x$ ;
- $\pi_1$ does not contain any definition of $x$.


The set of truly used variables at an edge $k = (\_, lab, v)$ is defined as:

| $lab$ | truly used |
|---|---|
| ; | $\emptyset$ |
| true $(e)$ | $Vars\,(e)$ |
| false $(e)$ | $Vars\,(e)$ |
| $x = e;$ | $Vars\,(e)$ $(*)$ |
| $x = M[e];$ | $Vars\,(e)$ $(*)$ |
| $M[e_1] = e_2;$ | $Vars(e_1) \cup Vars(e_2)$ |

$(*)$ – given that $x$ is truly live at $v$

# Example:



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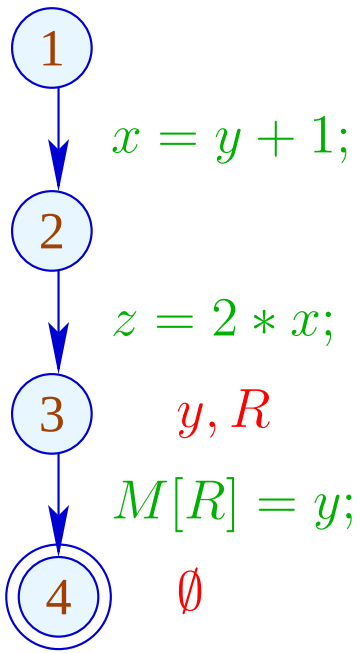

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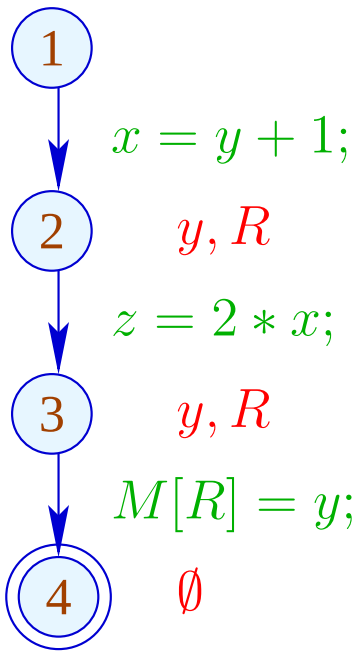

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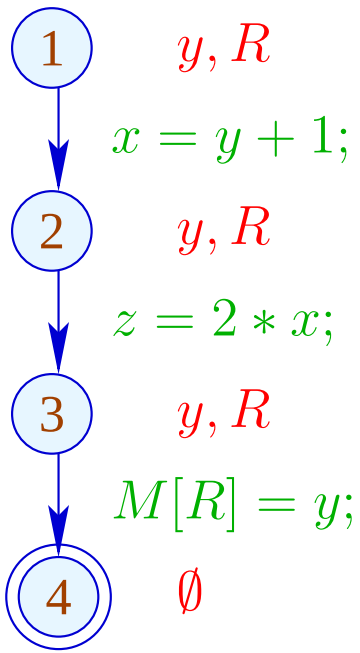

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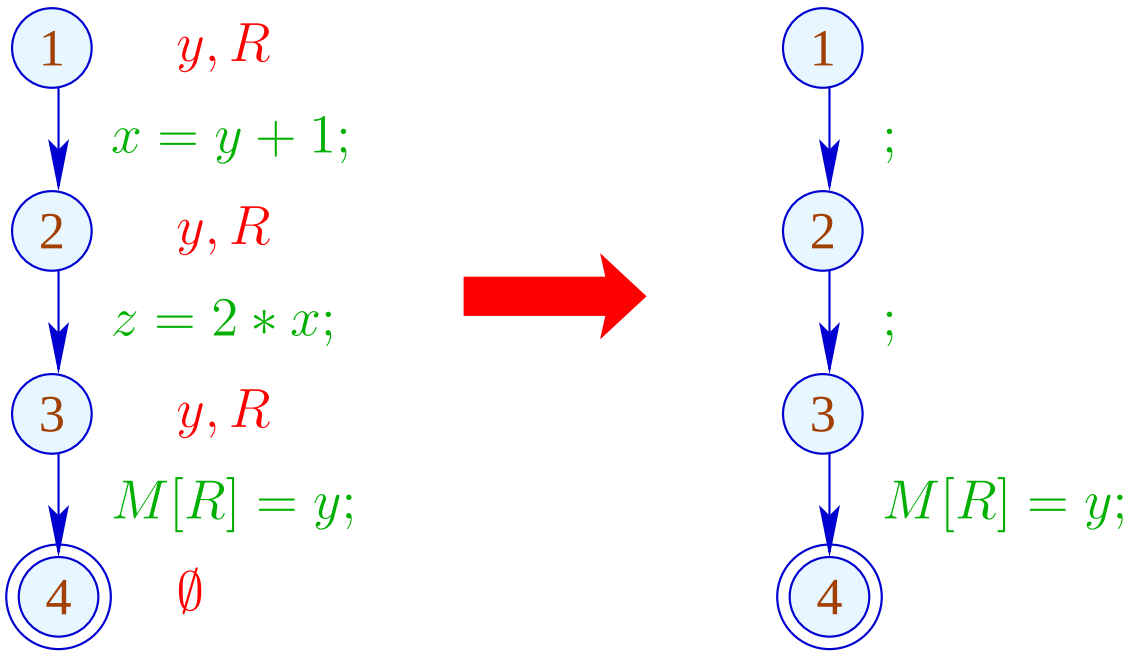

## The Effects of Edges:

$$
\begin{array}{lcl}
[\![;]\!]^\sharp\, L & = & L \\
[\![\mathrm{true}(e)]\!]^\sharp\, L & = & [\![\mathrm{false}(e)]\!]^\sharp\, L \;=\; L \cup \mathit{Vars}(e) \\
[\![x = e;]\!]^\sharp\, L & = & (L \backslash \{x\}) \cup \mathit{Vars}(e) \\
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[\![x = e;]\!]^\sharp\, L & = & (L \backslash \{x\}) \cup \; (x \in L)\,?\; \mathit{Vars}(e) : \emptyset \\
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## Note:

- The effects of edges for truly live variables are more complicated than for live variables
- Nonetheless, they are distributive !!

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  To see this, consider for $\mathbb{D} = 2^U$ , $f\, y = (u \in y)\,?\,b : \emptyset$ We verify:

$$\begin{aligned} f\,(y_1 \cup y_2) &= (u \in y_1 \cup y_2)\,?\,b : \emptyset \\ &= (u \in y_1 \vee u \in y_2)\,?\,b : \emptyset \\ &= (u \in y_1)\,?\,b : \emptyset \cup (u \in y_2)\,?\,b : \emptyset \\ &= f\, y_1 \cup f\, y_2 \end{aligned}$$

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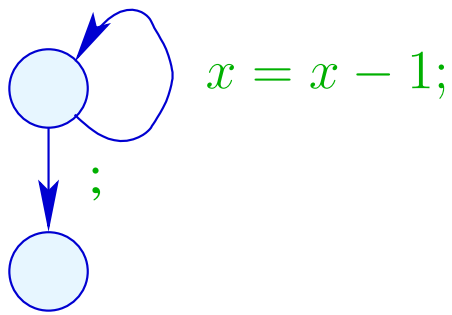
$\Longrightarrow$ the constraint system yields the MOP

- True liveness detects more superfluous assignments than repeated liveness !!!



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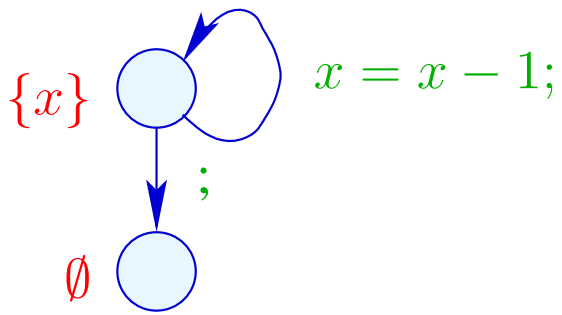
Liveness:



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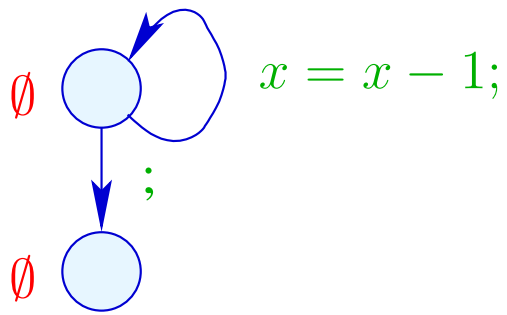
True Liveness:



# 7 Interval Analysis

Constant propagation attempts to determine values of variables.

However, variables may take on several values during program execution.

So, *the value* of a variable will often be unknown.

Next attempt: determine an interval enclosing all possible values that a variable may take on during program execution at a program point.

## Example:

```
for (i = 0; i < 42; i++)
    if (0 ≤ i ∧ i < 42){
        A_1 = A + i;
        M[A_1] = i;
    }
// A start address of an array
// if-statement does array-bounds check
```

Obviously, the inner check is superfluous.

Idea 1:

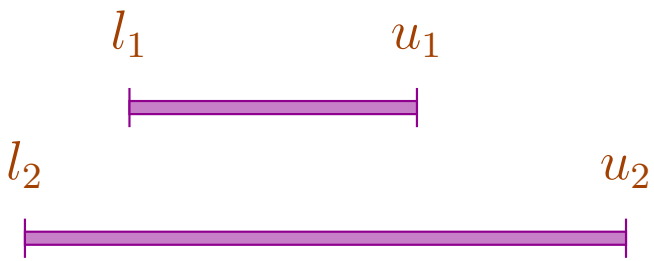
Determine for every variable $x$ the tightest possible interval of potential values.

Abstract domain:

$$\mathbb{I} = \{[l, u] \mid l \in \mathbb{Z} \cup \{-\infty\}, u \in \mathbb{Z} \cup \{+\infty\}, l \leq u\}$$

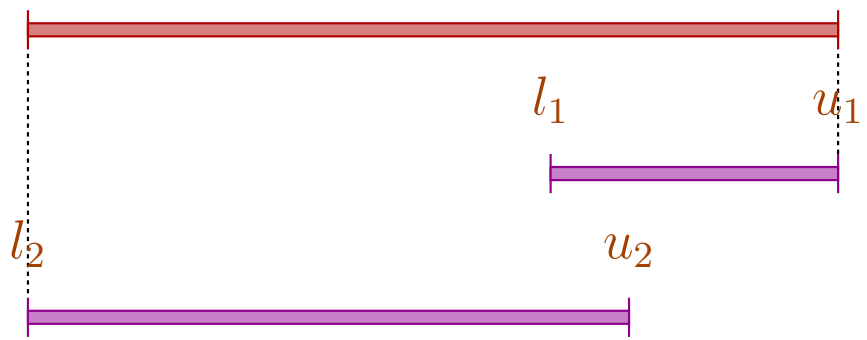
Partial order:

$$[l_1, u_1] \sqsubseteq [l_2, u_2] \qquad \text{iff} \qquad l_2 \leq l_1 \wedge u_1 \leq u_2$$



## Thus:

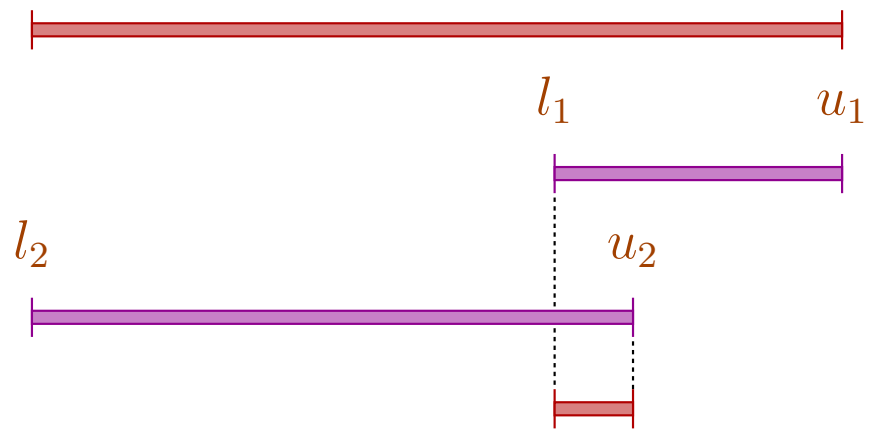
$$[l_1, u_1] \sqcup [l_2, u_2] \quad = \quad [l_1 \sqcap l_2, u_1 \sqcup u_2]$$



## Thus:

$$[l_1, u_1] \sqcup [l_2, u_2] = [l_1 \sqcap l_2, u_1 \sqcup u_2]$$

$$[l_1, u_1] \sqcap [l_2, u_2] = [l_1 \sqcup l_2, u_1 \sqcap u_2] \quad \text{whenever } (l_1 \sqcup l_2) \leq (u_1 \sqcap u_2)$$



## Caveat:

- $\rightarrow$ $\square$ is not a complete lattice,
- $\rightarrow$ $\square$ has infinite ascending chains, e.g.,

$$[0,0] \sqsubset [0,1] \sqsubset [-1,1] \sqsubset [-1,2] \sqsubset \ldots$$

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## Description Relation:

$$z \; \Delta \; [l,u] \qquad \text{iff} \qquad l \leq z \leq u$$

## Concretization:

$$\gamma\,[l,u] = \{z \in \mathbb{Z} \mid l \leq z \leq u\}$$

## Example:

$$\begin{aligned} \gamma\,[0,7] &= \{0,\ldots,7\} \\ \gamma\,[0,\infty] &= \{0,1,2,\ldots,\} \end{aligned}$$

## Computing with intervals: Interval Arithmetic.

### Addition:

$$\begin{aligned} [l_1,u_1] \;+^\sharp\; [l_2,u_2] &= [l_1+l_2,u_1+u_2] \qquad \text{where} \\ -\infty + \_ &= -\infty \\ +\infty + \_ &= +\infty \\ // \quad -\infty+\infty &\quad \text{cannot occur} \end{aligned}$$

Negation:

$$-^\sharp [l, u] \;=\; [-u, -l]$$

Multiplication:

$$\begin{aligned} [l_1, u_1] \;*^\sharp\; [l_2, u_2] &= [a, b] \qquad \text{where} \\ a &= l_1 l_2 \sqcap l_1 u_2 \sqcap u_1 l_2 \sqcap u_1 u_2 \\ b &= l_1 l_2 \sqcup l_1 u_2 \sqcup u_1 l_2 \sqcup u_1 u_2 \end{aligned}$$

Example:

$$\begin{aligned} [0, 2] \;*^\sharp\; [3, 4] &= [0, 8] \\ [-1, 2] \;*^\sharp\; [3, 4] &= [-4, 8] \\ [-1, 2] \;*^\sharp\; [-3, 4] &= [-6, 8] \\ [-1, 2] \;*^\sharp\; [-4, -3] &= [-8, 4] \end{aligned}$$

Division: $[l_1, u_1] /^\sharp [l_2, u_2] = [a, b]$

- If $0$ is not contained in the interval of the denominator, then:

$$\begin{aligned} a &= l_1/l_2 \sqcap l_1/u_2 \sqcap u_1/l_2 \sqcap u_1/u_2 \\ b &= l_1/l_2 \sqcup l_1/u_2 \sqcup u_1/l_2 \sqcup u_1/u_2 \end{aligned}$$

- If: $l_2 \leq 0 \leq u_2$ , we define:

$$[a, b] = [-\infty, +\infty]$$

Equality:

$$[l_1, u_1] ==^\sharp [l_2, u_2] \;=\; \begin{cases} true & \text{if} \quad l_1 = u_1 = l_2 = u_2 \\ false & \text{if} \quad u_1 < l_2 \vee u_2 < l_1 \\ \top & \text{otherwise} \end{cases}$$

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Example:

$$\begin{aligned} [42, 42] =\!=^\sharp [42, 42] &= true \\ [0, 7] =\!=^\sharp [0, 7] &= \top \\ [1, 2] =\!=^\sharp [3, 4] &= false \end{aligned}$$

Less:

$$[l_1, u_1] <^\sharp [l_2, u_2] = \begin{cases} true & \text{if} \quad u_1 < l_2 \\ false & \text{if} \quad u_2 \leq l_1 \\ \top & \text{otherwise} \end{cases}$$

Less:

$$[l_1, u_1] <^\sharp [l_2, u_2] = \begin{cases} true & \text{if} \quad u_1 < l_2 \\ false & \text{if} \quad u_2 \leq l_1 \\ \top & \text{otherwise} \end{cases}$$

Example:

$$\begin{aligned} [1, 2] <^\sharp [9, 42] &= true \\ [0, 7] <^\sharp [0, 7] &= \top \\ [3, 4] <^\sharp [1, 2] &= false \end{aligned}$$

By means of $\mathbb{I}$ we construct the complete lattice:

$$\mathbb{D}_{\mathbb{I}} = (\mathit{Vars} \to \mathbb{I})_{\perp}$$

Description Relation:

$$\rho \ \Delta \ D \qquad \text{iff} \qquad D \neq \perp \quad \wedge \quad \forall\, x \in \mathit{Vars} : \ (\rho\, x) \ \Delta \ (D\ x)$$

The abstract evaluation of expressions is defined analogously to constant propagation. We have:

$$([\![e]\!]\ \rho) \ \Delta \ ([\![e]\!]^{\sharp}\ D) \qquad \text{whenever} \qquad \rho \ \Delta \ D$$

# The Effects of Edges:

$$
\begin{array}{lcl}
[\![;]\!]^\sharp\, D & = & D \\
[\![x = e;]\!]^\sharp\, D & = & D \oplus \{x \mapsto [\![e]\!]^\sharp\, D\} \\
[\![x = M[e];]\!]^\sharp\, D & = & D \oplus \{x \mapsto \top\} \\
[\![M[e_1] = e_2;]\!]^\sharp\, D & = & D \\
[\![\text{true}\,(e)]\!]^\sharp\, D & = & \begin{cases} \bot & \text{if} \quad \text{definitely false} \\ D & \text{otherwise} \quad \text{possibly true} \end{cases} \\
[\![\text{false}\,(e)]\!]^\sharp\, D & = & \begin{cases} D & \text{if} \quad \text{possibly false} \\ \bot & \text{otherwise} \quad \text{definitely true} \end{cases}
\end{array}
$$

... given that $D \neq \bot$

## Better Exploitation of Conditions:

$$[\![\mathrm{Pos}\,(e)]\!]^\sharp\, D \;=\; \begin{cases} \bot & \text{if} \quad false = [\![e]\!]^\sharp\, D \\ D_1 & \text{otherwise} \end{cases}$$

where :

$$D_1 \;=\; \begin{cases} D \oplus \{x \mapsto (D\,x) \sqcap ([\![e_1]\!]^\sharp\, D)\} & \text{if } e \equiv x == e_1 \\ D \oplus \{x \mapsto (D\,x) \sqcap [-\infty, u]\} & \text{if } e \equiv x \leq e_1, [\![e_1]\!]^\sharp\, D = [\_, u] \\ D \oplus \{x \mapsto (D\,x) \sqcap [l, \infty]\} & \text{if } e \equiv x \geq e_1, [\![e_1]\!]^\sharp\, D = [l, \_] \end{cases}$$

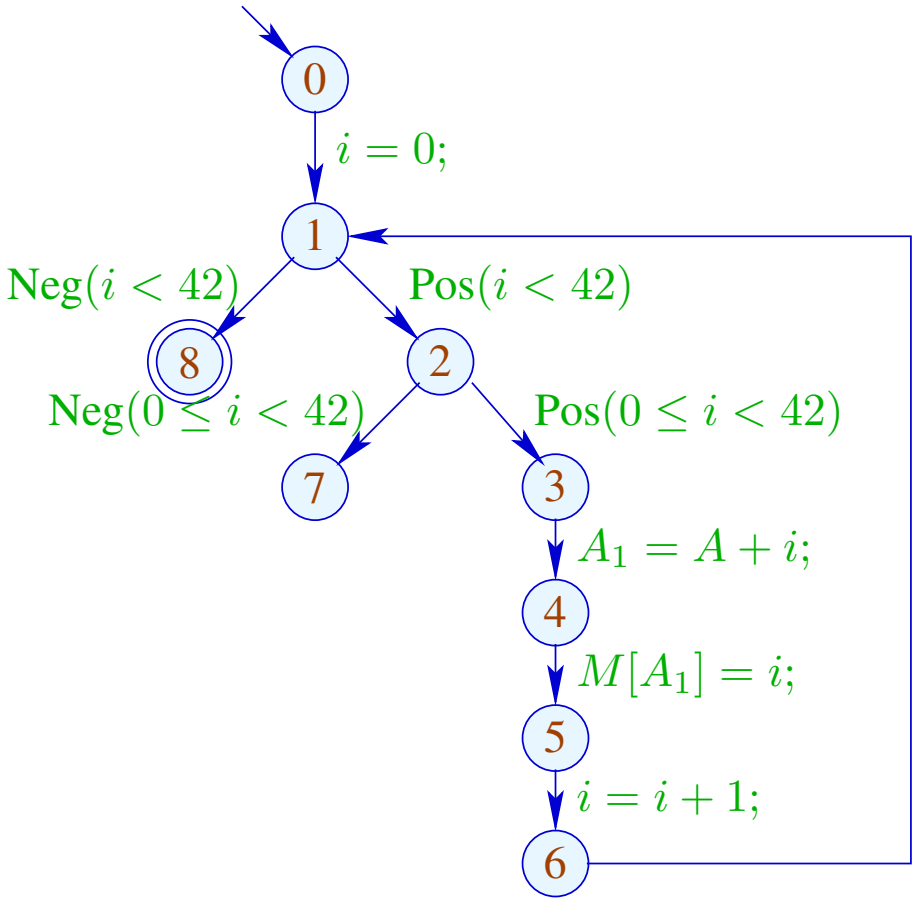
## Better Exploitation of Conditions (cont.):

$$[\![\text{Neg}\,(e)]\!]^\sharp\, D \;=\; \begin{cases} \bot & \text{if} \quad false \not\sqsubseteq [\![e]\!]^\sharp\, D \\ D_1 & \text{otherwise} \end{cases}$$

where :

$$D_1 \;=\; \begin{cases} D \oplus \{x \mapsto (D\,x) \sqcap ([\![e_1]\!]^\sharp\, D)\} & \text{if } e \equiv x \neq e_1 \\ D \oplus \{x \mapsto (D\,x) \sqcap [-\infty, u]\} & \text{if } e \equiv x > e_1, [\![e_1]\!]^\sharp\, D = [\_, u] \\ D \oplus \{x \mapsto (D\,x) \sqcap [l, \infty]\} & \text{if } e \equiv x < e_1, [\![e_1]\!]^\sharp\, D = [l, \_] \end{cases}$$

# Example:




| | $i$ | |
|---|---|---|
| | $l$ | $u$ |
| 0 | $-\infty$ | $+\infty$ |
| 1 | 0 | 42 |
| 2 | 0 | 41 |
| 3 | 0 | 41 |
| 4 | 0 | 41 |
| 5 | 0 | 41 |
| 6 | 1 | 42 |
| 7 | $\bot$ | |
| 8 | 42 | 42 |

## Problem:

→ The solution can be computed with RR-iteration — after about 42 rounds.

→ On some programs, iteration may never terminate.

## Idea: Widening

Accelerate the iteration — at the cost of precision

## Formalization of the Approach:

Let $x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n$ (1)

denote a system of constraints over $\mathbb{D}$

Define an accumulating iteration:

$$x_i = x_i \sqcup f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n \qquad (2)$$

We obviously have:

(a) $\underline{x}$ is a solution of (1) iff $\underline{x}$ is a solution of (2).

(b) The function $G : \mathbb{D}^n \to \mathbb{D}^n$ with
$G(x_1, \ldots, x_n) = (y_1, \ldots, y_n), \quad y_i = x_i \sqcup f_i(x_1, \ldots, x_n)$
is increasing, i.e., $\underline{x} \sqsubseteq G\,\underline{x}$ for all $\underline{x} \in \mathbb{D}^n$.

(c) The sequence $G^k \underline{\perp}$ , $k \geq 0$, is an ascending chain:

$$\underline{\perp} \sqsubseteq G\underline{\perp} \sqsubseteq \ldots \sqsubseteq G^k\underline{\perp} \sqsubseteq \ldots$$

(d) If $G^k \underline{\perp} = G^{k+1} \underline{\perp} = \underline{y}$ , then $\underline{y}$ is a solution of (1).

(e) If $\mathbb{D}$ has infinite strictly ascending chains, then (d) is not yet sufficient ...

but: we could consider the modified system of equations:

$$x_i = x_i \sqcup f_i(x_1, \ldots, x_n) \;, \quad i = 1, \ldots, n \tag{3}$$

for a binary operation widening:

$$\sqcup : \mathbb{D}^2 \to \mathbb{D} \qquad \text{with} \qquad v_1 \sqcup v_2 \sqsubseteq v_1 \sqcup v_2$$

(RR)-iteration for (3) still will compute a solution of (1)

## ... for Interval Analysis:

- The complete lattice is: $\mathbb{D}_{\mathbb{I}} = (Vars \to \mathbb{I})_\perp$
- the widening $\sqcup$ is defined by:

$$\perp \sqcup D = D \sqcup \perp = D \qquad \text{and for} \quad D_1 \neq \perp \neq D_2\text{:}$$

$$\begin{aligned}
(D_1 \sqcup D_2)\, x &= (D_1\, x) \sqcup (D_2\, x) \qquad \text{where} \\
[l_1, u_1] \sqcup [l_2, u_2] &= [l, u] \qquad \text{with} \\
l &= \begin{cases} l_1 & \text{if} \quad l_1 \leq l_2 \\ -\infty & \text{otherwise} \end{cases} \\
u &= \begin{cases} u_1 & \text{if} \quad u_1 \geq u_2 \\ +\infty & \text{otherwise} \end{cases}
\end{aligned}$$

$\Longrightarrow$ $\sqcup$ is not commutative !!!

## Example:

$$
\begin{aligned}
[0,2] \sqcup [1,2] &= [0,2] \\
[1,2] \sqcup [0,2] &= [-\infty,2] \\
[1,5] \sqcup [3,7] &= [1,+\infty]
\end{aligned}
$$

$\rightarrow$ Widening returns larger values more quickly.

$\rightarrow$ It should be constructed in such a way that termination of iteration is guaranteed.

$\rightarrow$ For interval analysis, widening bounds the number of iterations by:

$$\#points \cdot (1 + 2 \cdot \#Vars)$$

## Conclusion:

- In order to determine a solution of (1) over a complete lattice with infinite ascending chains, we define a suitable widening and then solve (3)

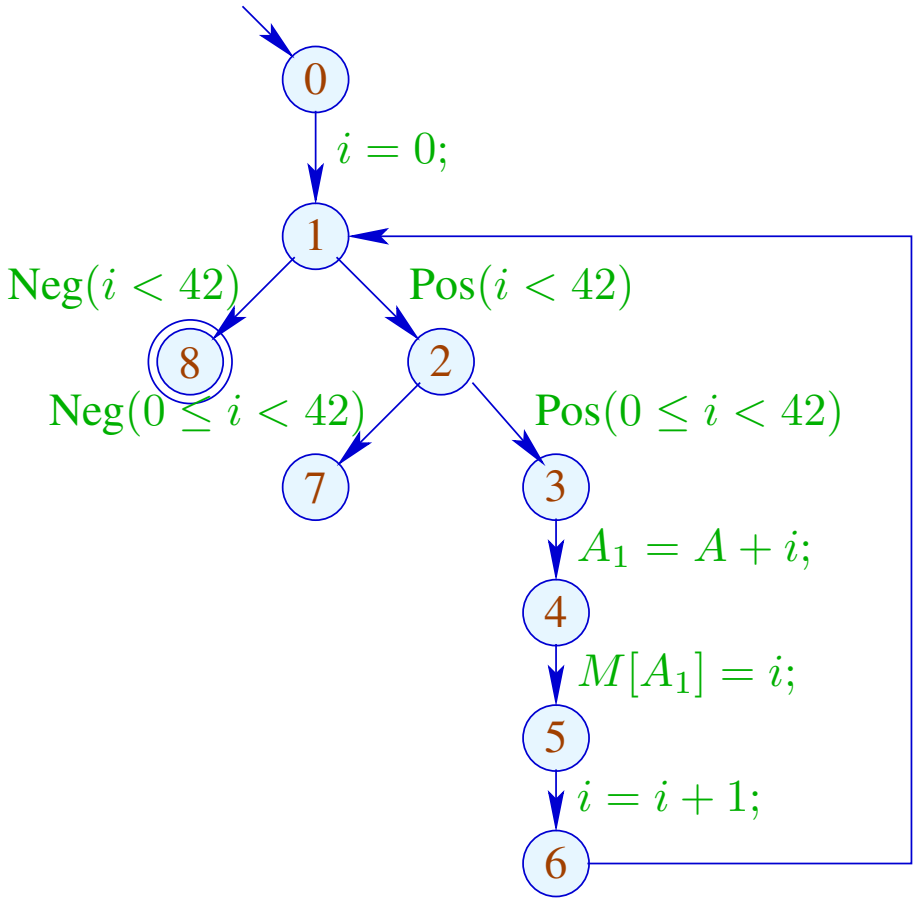
- Caveat: The construction of suitable widenings is a dark art !!!

  Often $\sqcup$ is chosen dynamically during iteration such that

  $\rightarrow$ the abstract values do not get too complicated;

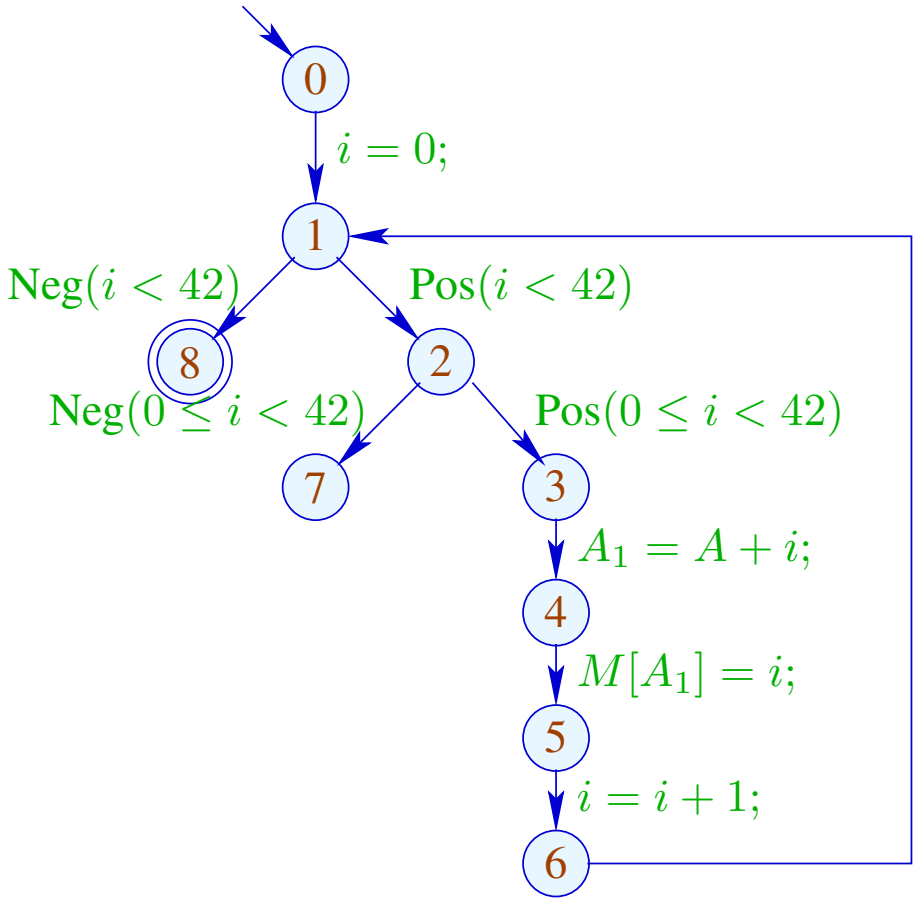
  $\rightarrow$ the number of updates remains bounded ...

# Our Example:




| | 1 | |
|---|---|---|
| | $l$ | $u$ |
| 0 | $-\infty$ | $+\infty$ |
| 1 | 0 | 0 |
| 2 | 0 | 0 |
| 3 | 0 | 0 |
| 4 | 0 | 0 |
| 5 | 0 | 0 |
| 6 | 1 | 1 |
| 7 | $\bot$ | |
| 8 | $\bot$ | |

# Our Example:




| | 1 | | 2 | | 3 | |
|---|---|---|---|---|---|---|
| | $l$ | $u$ | $l$ | $u$ | $l$ | $u$ |
| 0 | $-\infty$ | $+\infty$ | $-\infty$ | $+\infty$ | | |
| 1 | 0 | 0 | 0 | $+\infty$ | | |
| 2 | 0 | 0 | 0 | $+\infty$ | | |
| 3 | 0 | 0 | 0 | $+\infty$ | | |
| 4 | 0 | 0 | 0 | $+\infty$ | dito | |
| 5 | 0 | 0 | 0 | $+\infty$ | | |
| 6 | 1 | 1 | 1 | $+\infty$ | | |
| 7 | $\bot$ | | 42 | $+\infty$ | | |
| 8 | $\bot$ | | 42 | $+\infty$ | | |

... obviously, the result is disappointing.

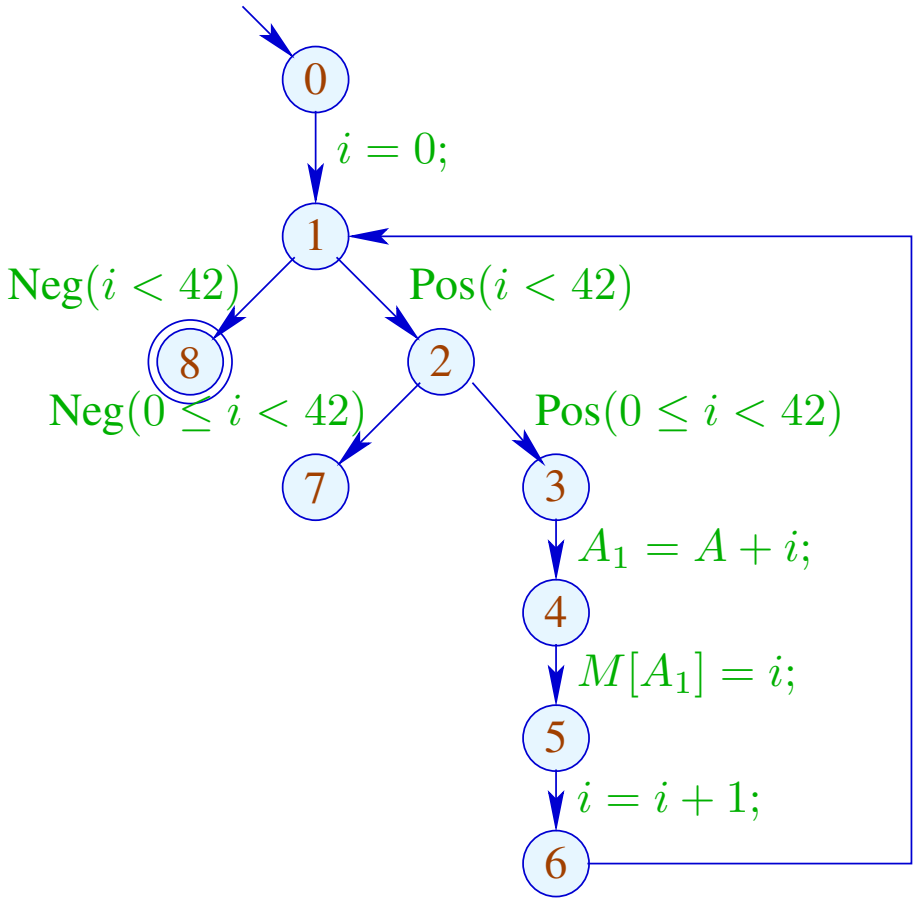
## Idea 2:

In fact, acceleration with $\sqcup$ need only be applied at sufficiently many places!

A set $I$ is a loop separator, if every loop contains at least one point from $I$

If we apply widening only at program points from such a set $I$, then RR-iteration still terminates !!!

# In our Example:

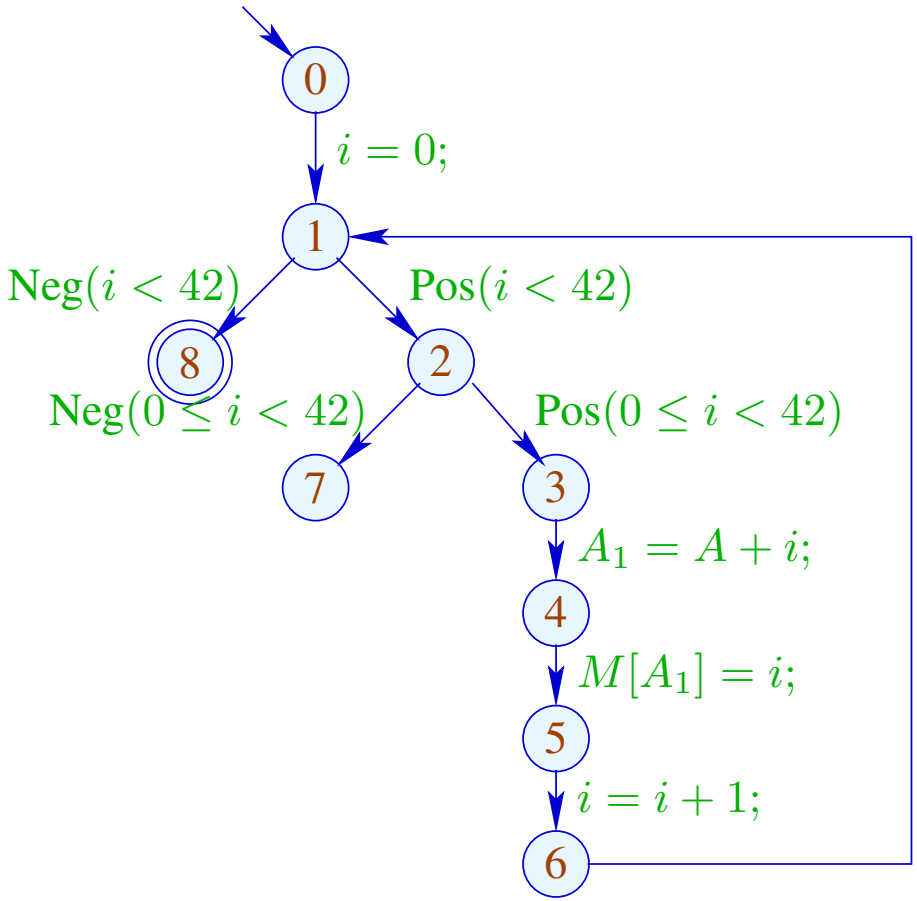



$I_1 = \{1\}$ or:

$I_2 = \{2\}$ or:

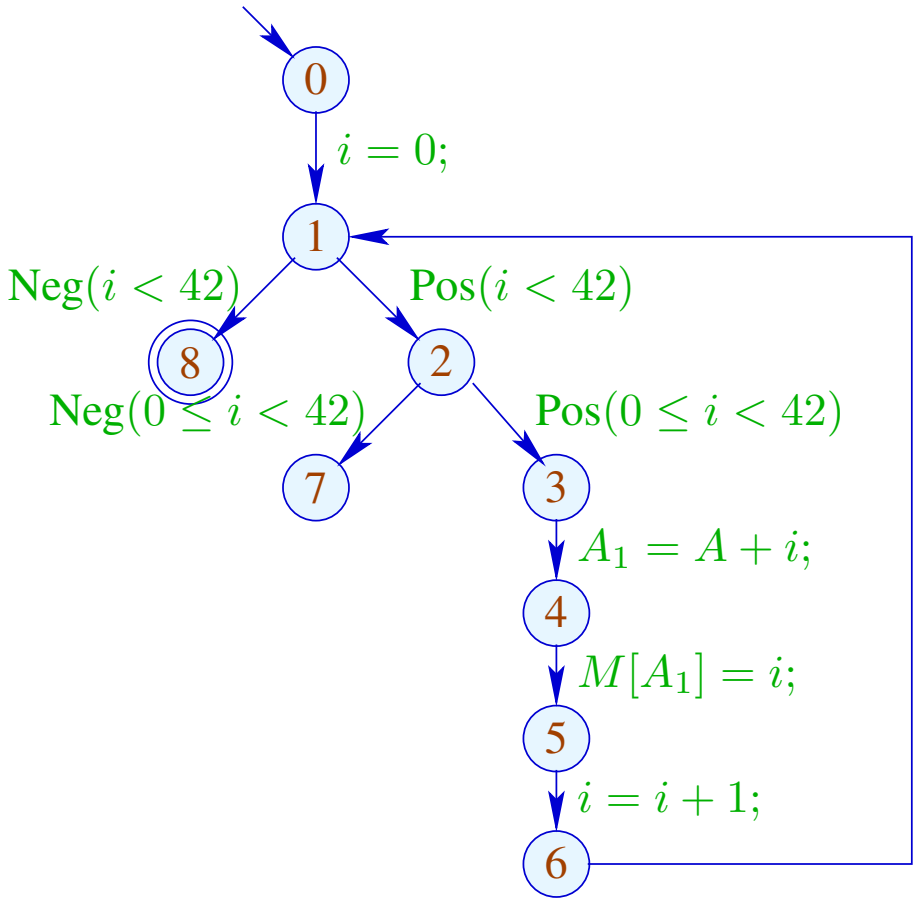
$I_3 = \{3\}$

## The Analysis with $I = \{1\}$ :




| | 1 | | 2 | | 3 | |
|---|---|---|---|---|---|---|
| | $l$ | $u$ | $l$ | $u$ | $l$ | $u$ |
| 0 | $-\infty$ | $+\infty$ | $-\infty$ | $+\infty$ | | |
| 1 | 0 | 0 | 0 | $+\infty$ | | |
| 2 | 0 | 0 | 0 | 41 | | |
| 3 | 0 | 0 | 0 | 41 | | |
| 4 | 0 | 0 | 0 | 41 | dito | |
| 5 | 0 | 0 | 0 | 41 | | |
| 6 | 1 | 1 | 1 | 42 | | |
| 7 | $\bot$ | | $\bot$ | | | |
| 8 | $\bot$ | | 42 | $+\infty$ | | |

## The Analysis with $I = \{2\}$ :




| | 1 | | 2 | | 3 | | 4 |
|---|---|---|---|---|---|---|---|
| | $l$ | $u$ | $l$ | $u$ | $l$ | $u$ | |
| 0 | $-\infty$ | $+\infty$ | $-\infty$ | $+\infty$ | $-\infty$ | $+\infty$ | |
| 1 | 0 | 0 | 0 | 1 | 0 | 42 | |
| 2 | 0 | 0 | 0 | $+\infty$ | 0 | $+\infty$ | |
| 3 | 0 | 0 | 0 | 41 | 0 | 41 | |
| 4 | 0 | 0 | 0 | 41 | 0 | 41 | dito |
| 5 | 0 | 0 | 0 | 41 | 0 | 41 | |
| 6 | 1 | 1 | 1 | 42 | 1 | 42 | |
| 7 | $\bot$ | | 42 | $+\infty$ | 42 | $+\infty$ | |
| 8 | $\bot$ | | $\bot$ | | 42 | 42 | |

## Discussion:

- Both runs of the analysis determine interesting information,
- The run with $I = \{2\}$ proves that always $i = 42$ after leaving the loop.
- Only the run with $I = \{1\}$ finds, however, that the outer check makes the inner check superfluous.

## How can we find a suitable loop separator $I$ ???

## Idea 3: Narrowing

Let $\underline{x}$ denote any solution of (1), i.e.,

$$x_i \sqsupseteq f_i\,\underline{x}\ , \qquad i = 1, \ldots, n$$

Then for monotonic $f_i$,

$$\underline{x} \sqsupseteq F\,\underline{x} \sqsupseteq F^2\,\underline{x} \sqsupseteq \ldots \sqsupseteq F^k\,\underline{x} \sqsupseteq \ldots$$

// Narrowing Iteration

## Idea 3: Narrowing

Let $\underline{x}$ denote any solution of (1), i.e.,

$$x_i \sqsupseteq f_i\,\underline{x}\ , \qquad i = 1, \ldots, n$$

Then for monotonic $f_i$,

$$\underline{x} \sqsupseteq F\,\underline{x} \sqsupseteq F^2\,\underline{x} \sqsupseteq \ldots \sqsupseteq F^k\,\underline{x} \sqsupseteq \ldots$$

// Narrowing Iteration

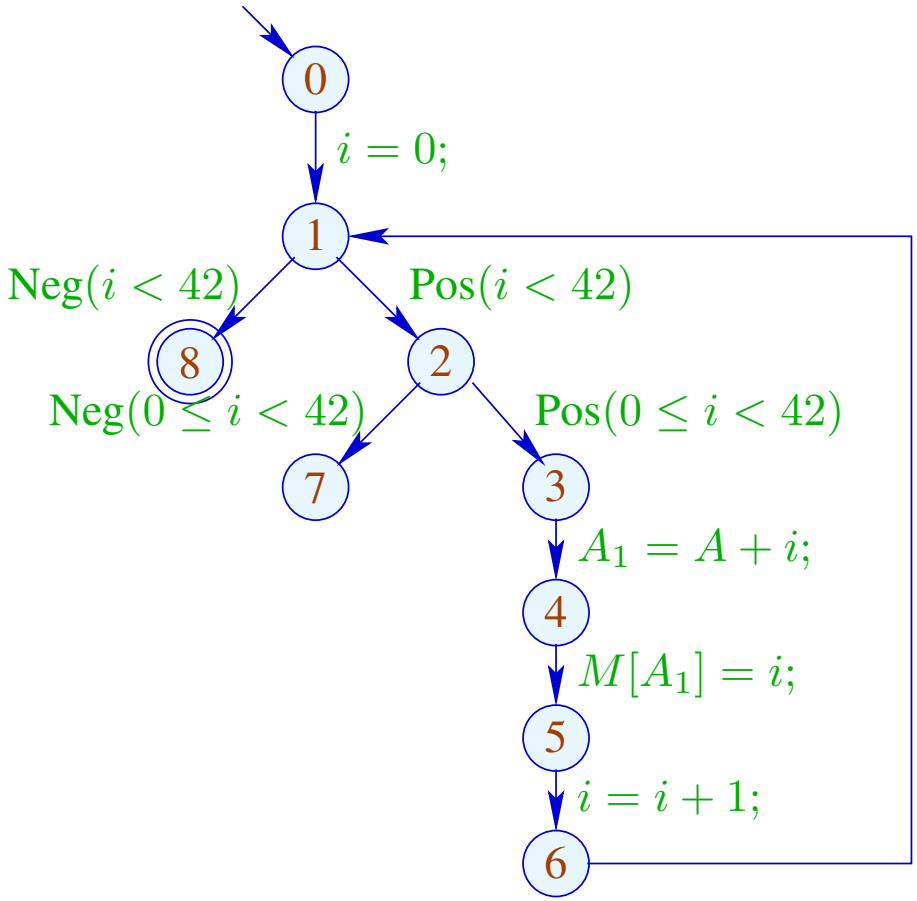
Every tuple $F^k\,\underline{x}$ is a solution of (1)

$\Longrightarrow$

Termination is no problem anymore:
we stop whenever we want

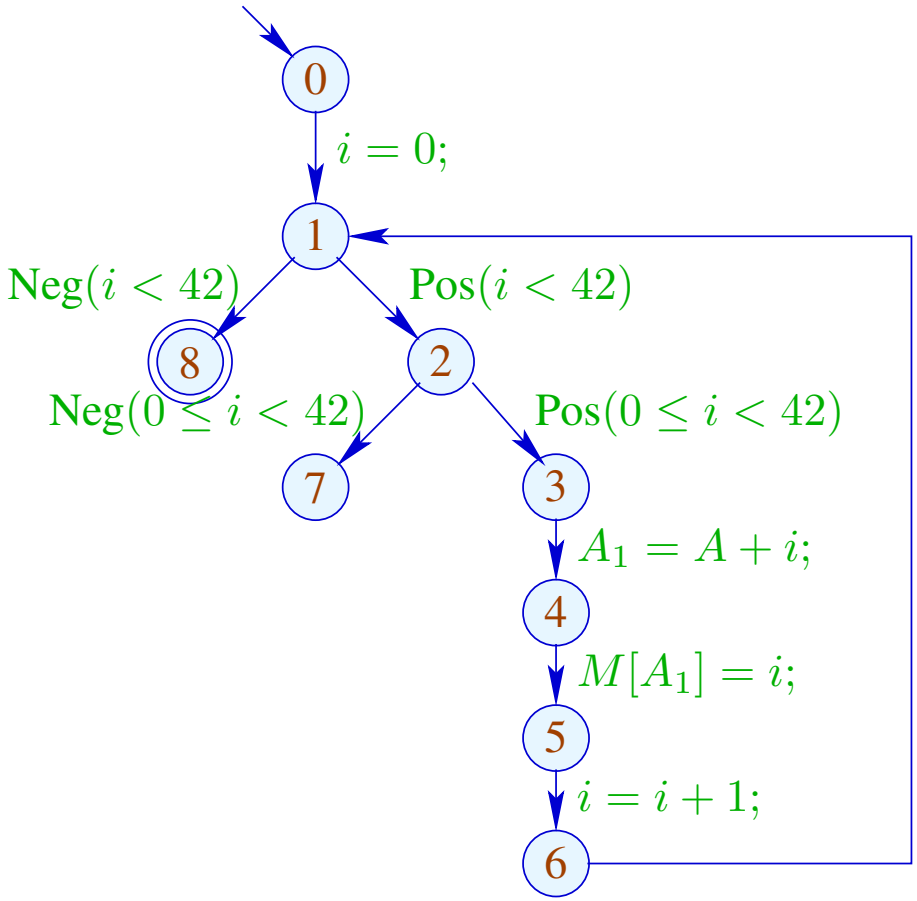
// The same also holds for RR-iteration.

# Narrowing Iteration in the Example:



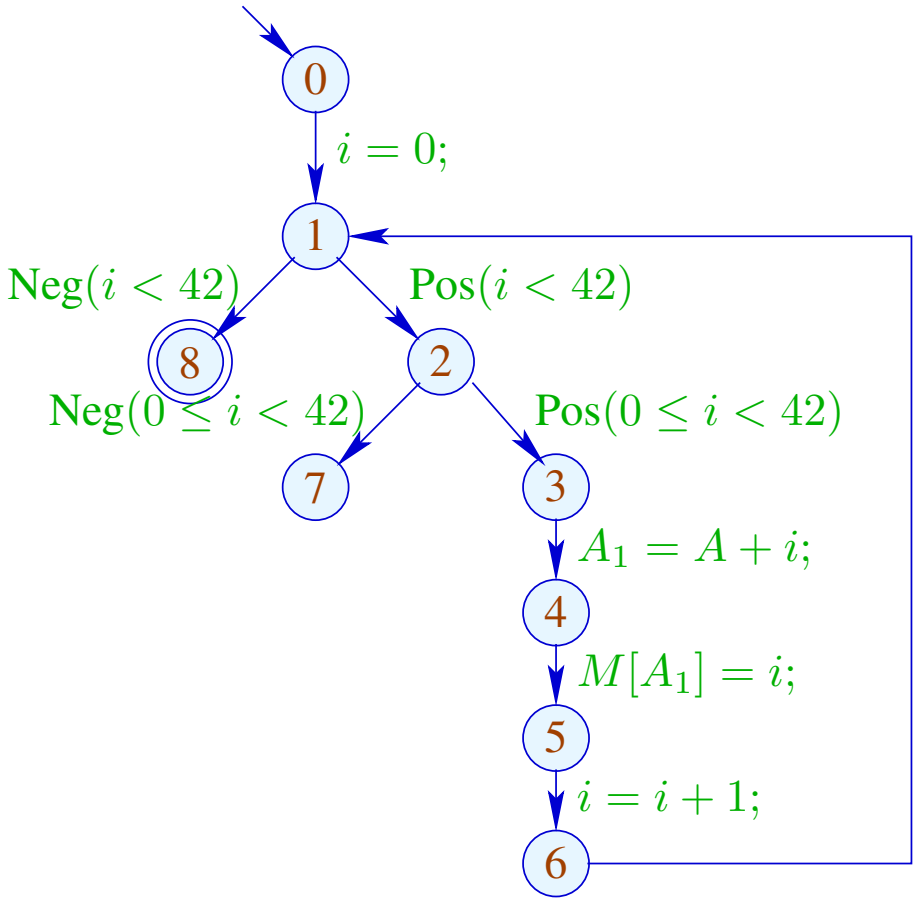

| | 0 | |
|---|---|---|
| | $l$ | $u$ |
| 0 | $-\infty$ | $+\infty$ |
| 1 | 0 | $+\infty$ |
| 2 | 0 | $+\infty$ |
| 3 | 0 | $+\infty$ |
| 4 | 0 | $+\infty$ |
| 5 | 0 | $+\infty$ |
| 6 | 1 | $+\infty$ |
| 7 | 42 | $+\infty$ |
| 8 | 42 | $+\infty$ |

# Narrowing Iteration in the Example:




| | 0 | | 1 | |
|---|---|---|---|---|
| | $l$ | $u$ | $l$ | $u$ |
| 0 | $-\infty$ | $+\infty$ | $-\infty$ | $+\infty$ |
| 1 | 0 | $+\infty$ | 0 | $+\infty$ |
| 2 | 0 | $+\infty$ | 0 | 41 |
| 3 | 0 | $+\infty$ | 0 | 41 |
| 4 | 0 | $+\infty$ | 0 | 41 |
| 5 | 0 | $+\infty$ | 0 | 41 |
| 6 | 1 | $+\infty$ | 1 | 42 |
| 7 | 42 | $+\infty$ | $\bot$ | |
| 8 | 42 | $+\infty$ | 42 | $+\infty$ |

# Narrowing Iteration in the Example:




| | 0 | | 1 | | 2 | |
|---|---|---|---|---|---|---|
| | $l$ | $u$ | $l$ | $u$ | $l$ | $u$ |
| 0 | $-\infty$ | $+\infty$ | $-\infty$ | $+\infty$ | $-\infty$ | $+\infty$ |
| 1 | 0 | $+\infty$ | 0 | $+\infty$ | 0 | 42 |
| 2 | 0 | $+\infty$ | 0 | 41 | 0 | 41 |
| 3 | 0 | $+\infty$ | 0 | 41 | 0 | 41 |
| 4 | 0 | $+\infty$ | 0 | 41 | 0 | 41 |
| 5 | 0 | $+\infty$ | 0 | 41 | 0 | 41 |
| 6 | 1 | $+\infty$ | 1 | 42 | 1 | 42 |
| 7 | 42 | $+\infty$ | $\bot$ | | $\bot$ | |
| 8 | 42 | $+\infty$ | 42 | $+\infty$ | 42 | 42 |

## Discussion:

- $\rightarrow$ We start with a safe approximation.
- $\rightarrow$ We find that the inner check is redundant :-)
- $\rightarrow$ We find that at exit from the loop, always $i = 42$
- $\rightarrow$ It was not necessary to construct an optimal loop separator

## Last Question: