Global Value Numbering

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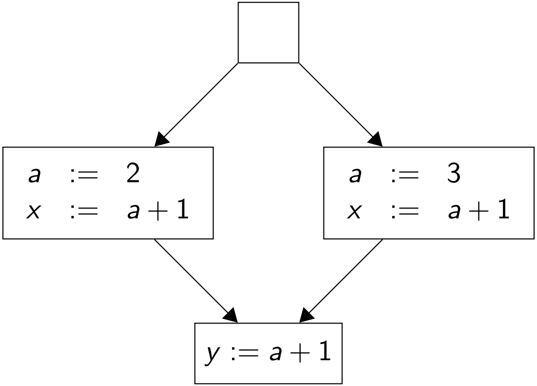
Value Numbering



- Replace second computation of $a + 1$ with a copy from x

Value Numbering

- Goal: Eliminate redundant computations
- Find out if two variables have the same value at given program point
  - In general undecidable
- Potentially replace computation of latter variable with contents of the former
- Resort to Herbrand equivalence:
  - Do not consider the interpretation of operators
  - Two expressions are equal if they are structurally equal
- This lecture: A costly program analysis which finds all Herbrand equivalences in a program and a “light-weight” version that is often used in practice.

Herbrand Interpretation

- The Herbrand interpretation $\mathcal{I}$ of an n-ary operator ω is given as

$$\mathcal{I}(\omega) : T^n \to T \qquad \mathcal{I}(\omega)(t_1, \dots, t_n) := \omega(t_1, \dots, t_n)$$

  Especially, constants are mapped to themselves

- With a state σ that maps variables to terms

$$\sigma : V \to T$$

- we can define the Herbrand semantics $\langle t \rangle \sigma$ of a term t

$$\langle t \rangle \sigma := \begin{cases} \sigma(v) & \text{if } t = v \text{ is a variable} \\ \mathcal{I}(\omega)(\langle x_1 \rangle \sigma, \dots, \langle x_n \rangle \sigma) & \text{if } t = \omega(x_1, \dots, x_n) \end{cases}$$

Programs with Herbrand Semantics

- We now interpret the program with respect to the Herbrand semantics
- For an assignment

$$x \leftarrow t$$

  the semantics is defined by:

$$[\![x \leftarrow t]\!]\sigma := \sigma \left[\langle t \rangle \sigma / x\right]$$

- The state after executing a path $p : \ell_1, \ldots, \ell_n$ starting with state σ_0 is then:

$$[\![p]\!]\sigma_0 := ([\![\ell_n]\!] \circ \cdots \circ [\![\ell_1]\!])\sigma_0$$

- Two expressions t_1 and t_2 are Herbrand equivalent at a program point ℓ iff

$$\forall p : r, \ldots, \ell. \, \langle t_1 \rangle [\![p]\!]\sigma_0 = \langle t_2 \rangle [\![p]\!]\sigma_0$$

Kildall's Analysis

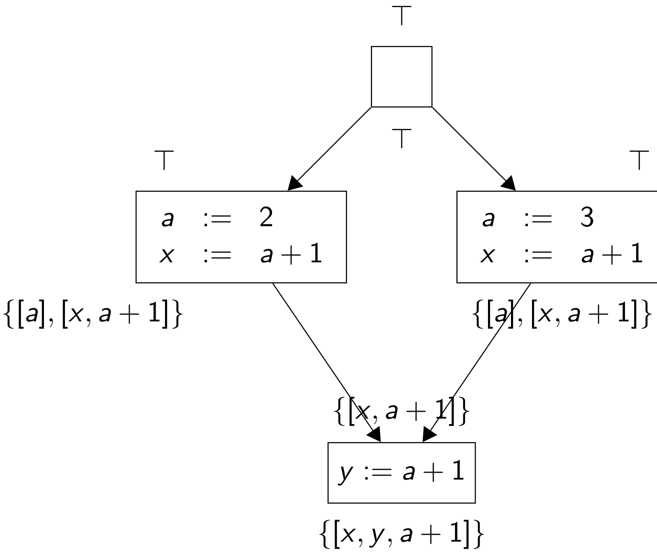
- Track Herbrand equivalences with a forward data flow analysis
- A lattice element is an equivalence class of the terms and variables of the program
- The equivalence relation is a congruence relation w.r.t. to the operators in our expression language.
  For each operator ω, each eq. relation R, and $e, e_1, \dots \in V \cup T$:

$$e \; R \; (e_1 \; \omega \; e_2) \implies e_1 \; R \; e_1' \implies e_2 \; R \; e_2' \implies e \; R \; (e_1' \; \omega \; e_2')$$

- Two equivalence classes are joined by intersecting them
  $R \sqcup S := R \cap S := \{(a, b) \mid a \; R \; b \wedge a \; S \; b\}$
- $\bot = \{(x, y) \mid x, y \in V \cup T\}$
- $\top = \{(x, x) \mid x \in V \cup T\}$

Kildall's Analysis

Example



Kildall's Analysis

Transfer Functions

... of an assignment

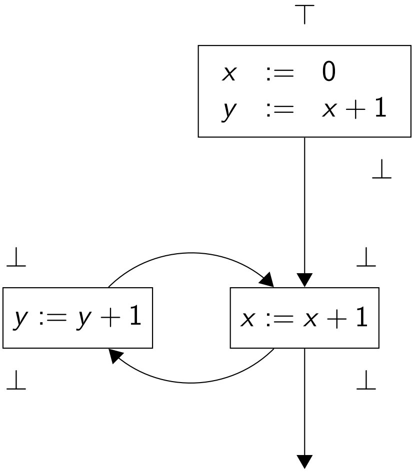
$$\ell : x \leftarrow t$$

- Compute a new partition checking (in the old partition) who is equivalent if we replace x by t

$$[\![x \leftarrow t]\!]^\sharp R := \{(t_1, t_2) \mid t_1[t/x] \; R \; t_2[t/x]\}$$

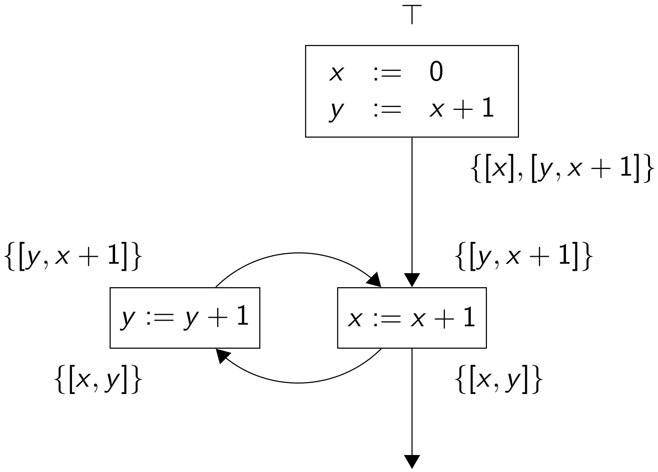
Kildall's Analysis

Example



Kildall's Analysis

Example



Kildall's Analysis

Comments

- Kildall's Analysis is sound and complete
  it discovers all Herbrand equivalences in the program

- Naïve implementations suffer from exponential explosion (pathological):
  - Because the equivalence relation must be congruence, size of eq. classes can explode:

$$R = \{[a, b], [c, d], [e, f], [x, a + c, a + d, b + c, b + d],$$
$$[y, x + e, x + f, (a + c) + e, \ldots, (b + d) + f]\}$$

- In practice: Use value graph.
  Do not make congruence explicit in representation.

- Theoretical results (Gulwani & Necula 2004):
  - Even in acyclic programs, detecting all equivalences can lead to exponential-sized value graphs
  - Detecting only equivalences among terms in the program is polynomial (linear-sized representation of equivalences per program point)

Strong Equivalence DAGs (SED)

A SED G is a DAG (N, E). Let N be the set of nodes of the graph. Every node n is a pair (V, t) of a set of variables and a type[1]

$$t ::= \bot \mid c \mid \oplus(n_1, \ldots, n_k)$$

A type $\oplus(n_1, \ldots, n_k)$ indicates, that

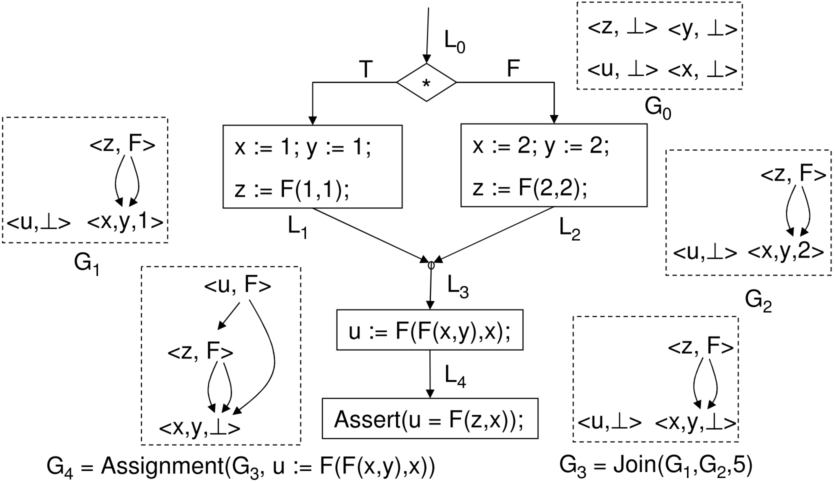
$$\{(n, n_1), \ldots, (n, n_k)\} \in E$$

A node $n = (V, t)$ in the SED stands for a set of terms $T(n)$

$$\begin{aligned} T((V, \bot)) &= V \\ T((V, c)) &= V \cup \{c\} \\ T((V, \oplus(n_1, \ldots, n_k))) &= V \cup \{\oplus(e_1, \ldots, e_k) \mid e_i \in T(n_i)\} \end{aligned}$$

[1]Note that $\bot$ does not denote the "empty set of states" here

Strong Equivalence DAGs (SED)



From: Gulwani & Necula. A Polynomial-Time Algorithm for Global Value Numbering. SAS 2004

The Alpern, Wegman, Zadeck (AWZ) Algorithm

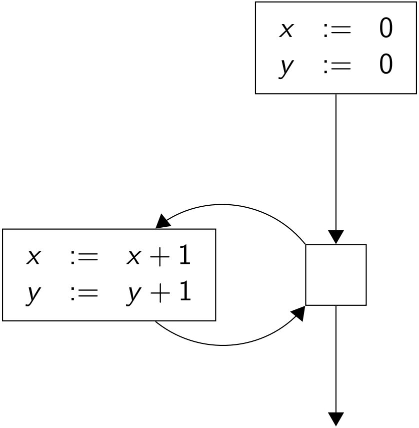
- Incomplete
- Flow-insensitive
  - does not compute the equivalences for every program point but sound equivalences for the whole program
- Uses SSA
  - Control-flow joins are represented by ϕs
  - Treat ϕs like every other operator (cause for incompleteness)
  - Source of imprecision
- Interpret the SSA data dependence graph as a finite automaton and minimize it
  - Refine partitions of "equivalent states"
  - Using Hopcroft's algorithm, this can be done in $O(e \cdot \log e)$

The AWZ Algorithm

- In contrast to finite automata, do not create two partitions but a class for every operator symbol
  - Note that the ϕ's block is part of the operator
  - Two ϕs from different blocks have to be in different classes
- Optimistically place all nodes with the same operator symbol in the same class
  - Finds the least fixpoint
  - You can also start with singleton classes and merge but this will (in general) not give the least fixpoint
- Successively split class when two nodes in the class are detected not equivalent

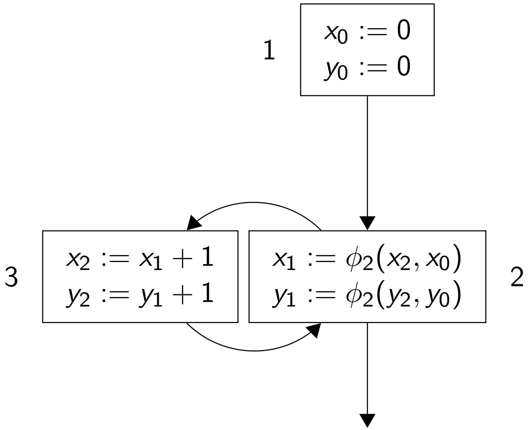
The AWZ Algorithm

Example



The AWZ Algorithm

Example



The AWZ Algorithm

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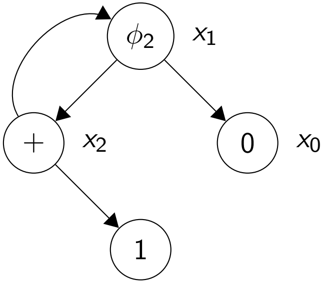

The AWZ Algorithm

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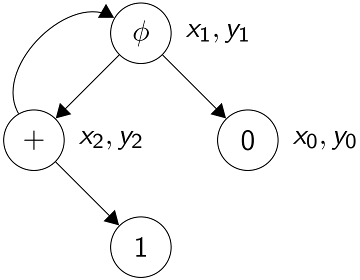
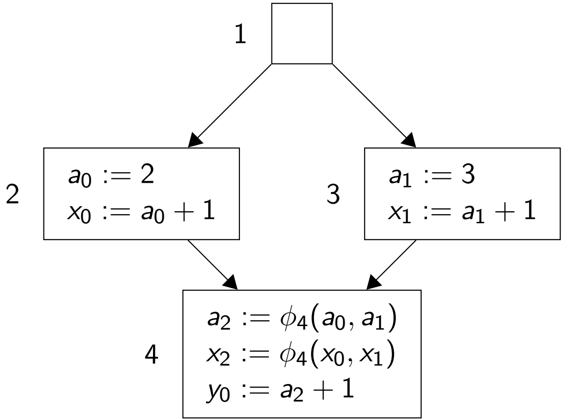
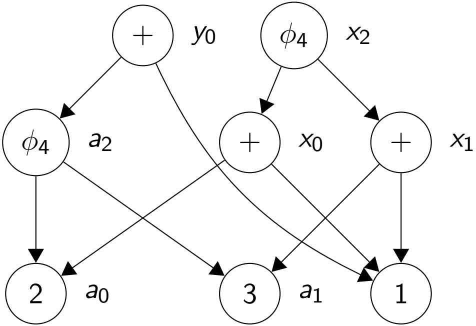


Kildall compared to AWZ



Kildall compared to AWZ



Kildall compared to AWZ