Static Program Analysis

Seidl/Wilhelm/Hack: Compiler Design – Analysis and Transformation, Springer Verlag, 2012

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A Short History of Static Program Analysis

- Early high-level programming languages were implemented on very small and very slow machines.
- Compilers needed to generate executables that were extremely efficient in space and time.
- Compiler writers invented efficiency-increasing program transformations, wrongly called optimizing transformations.
- Transformations must not change the semantics of programs.
- Enabling conditions guaranteed semantics preservation.
- Enabling conditions were checked by static analysis of programs.

Theoretical Foundations of Static Program Analysis

- Theoretical foundations for the solution of recursive equations: Kleene (1930s), Tarski (1955)
- Gary Kildall (1972) clarified the lattice-theoretic foundation of data-flow analysis.
- Patrick Cousot (1974) established the relation to the programming-language semantics.

Static Program Analysis as a Verification Method

- Automatic method to derive invariants about program behavior, answers questions about program behavior:
  - will index always be within bounds at program point p?
  - will memory access at p always hit the cache?
- answers of sound static analysis are correct, but approximate: don't know is a valid answer!
- analyses proved correct wrt. language semantics,

Proposed Lectures Content:

1. Introductory example: rules-of-sign analysis
2. theoretical foundations: lattices
3. an operational semantics of the language
4. another example: constant propagation
5. relating the semantics to the analysis—correctness proofs
6. some further static analyses in compilers: Elimination of superfluous computations
   - $\rightarrow$ available expressions
   - $\rightarrow$ live variables
   - $\rightarrow$ array-bounds checks

1 Introduction

... in this course and in the Seidl/Wilhelm/Hack book:

a simple imperative programming language with:

- variables // registers
- $R = e;$ // assignments
- $R = M[e];$ // loads
- $M[e_1] = e_2;$ // stores
- if (e) s_1 else s_2 // conditional branching
- goto $L;$ // no loops

Intermediate language into which (almost) everything can be compiled.

However, no procedures. So, only intra-procedural analyses!

2 Example: Rules-of-Sign Analysis

Starting Point: Questions about a program, mostly at a particular program point:

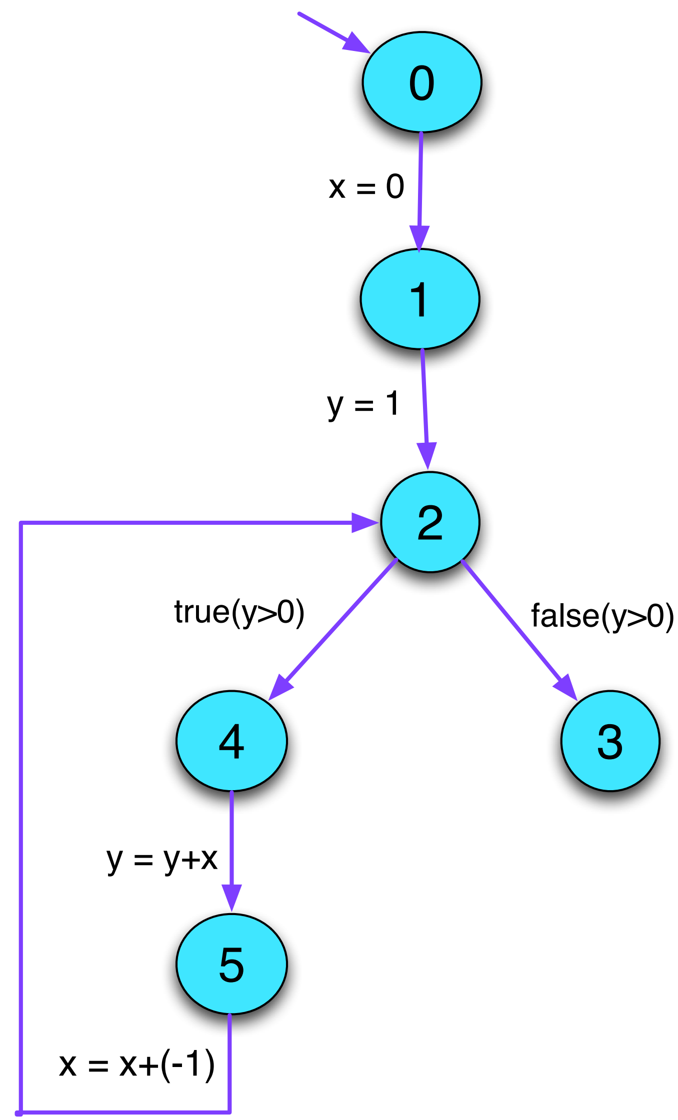
- May variable x have value 0 when program execution reaches this program point? $\longrightarrow$ Attempt to exclude division by 0.
- May x have a negative value? $\longrightarrow$ Attempt to exclude sqrt of a negative number.

Solution: Determine at each program point the sign of the values of all variables of numeric type.

Determines a sound, but maybe approximate answer.

Example program represented as *control-flow graph*

```
x = 0;
y = 1;
while (y > 0) do
      y = y + x;
      x = x + (-1);
```



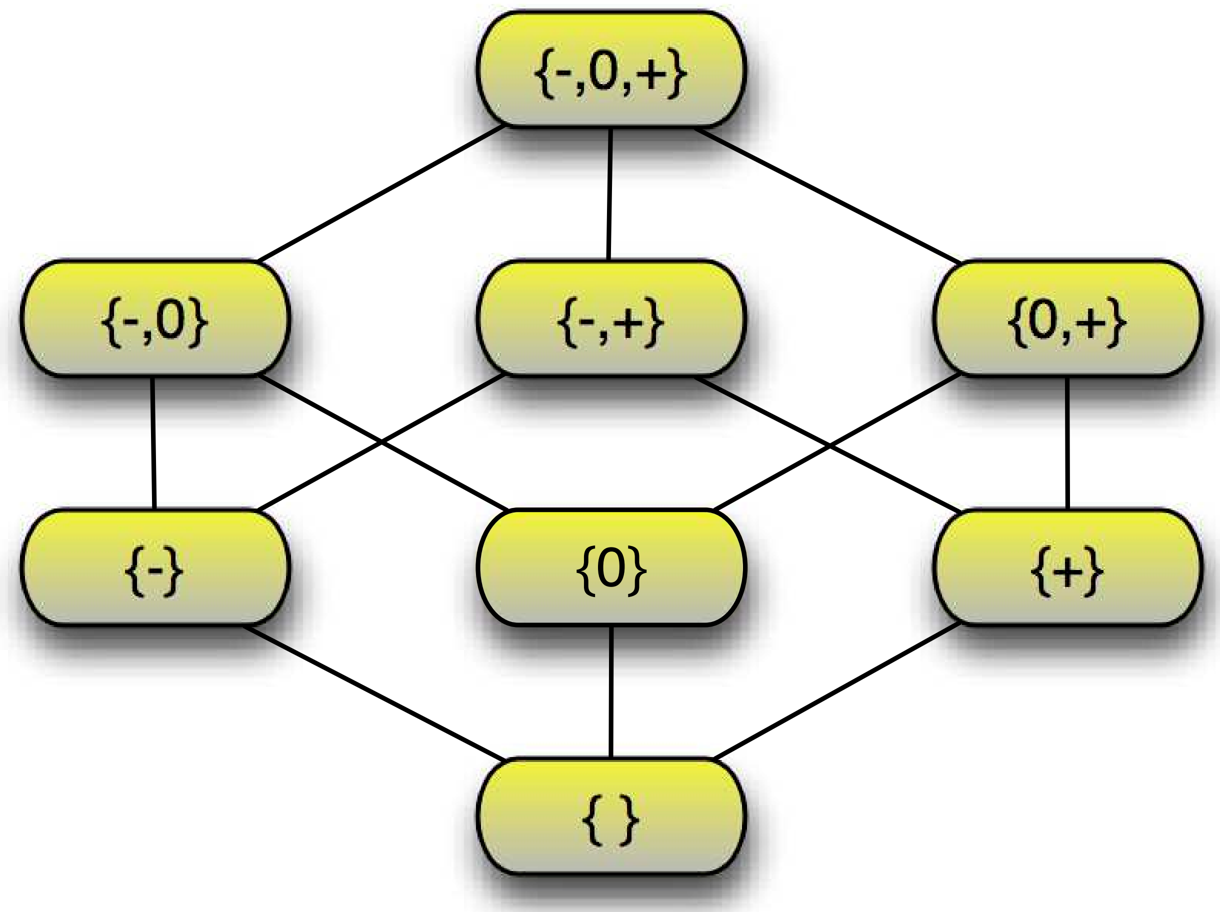
What are the ingredients that we need?

More ingredients?

All the ingredients:

- a set of information elements, each a set of possible signs,
- a partial order, “$\sqsubseteq$”, on these elements, specifying the ”relative strength” of two information elements,
- these together form the abstract domain, a lattice,
- functions describing how signs of variables change by the execution of a statement, abstract edge effects,
- these need an abstract arithmetic, an arithmetic on signs.

We construct the abstract domain for single variables starting with the lattice $Signs = 2^{\{-,0,+\}}$ with the relation "$\sqsubseteq$" ="$\supseteq$".



The analysis should ”bind” program variables to elements in $Signs$.

So, the abstract domain is $\mathbb{D} = (\mathit{Vars} \to \mathit{Signs})_\perp$, a Sign-environment.

$\perp \in \mathbb{D}$ is the function mapping all arguments to $\{\}$.

The partial order on $\mathbb{D}$ is $\quad D_1 \sqsubseteq D_2 \quad$ iff

$$D_1 = \perp \qquad \text{or}$$

$$D_1\, x \supseteq D_2\, x \qquad (x \in \mathit{Vars})$$

Intuition?

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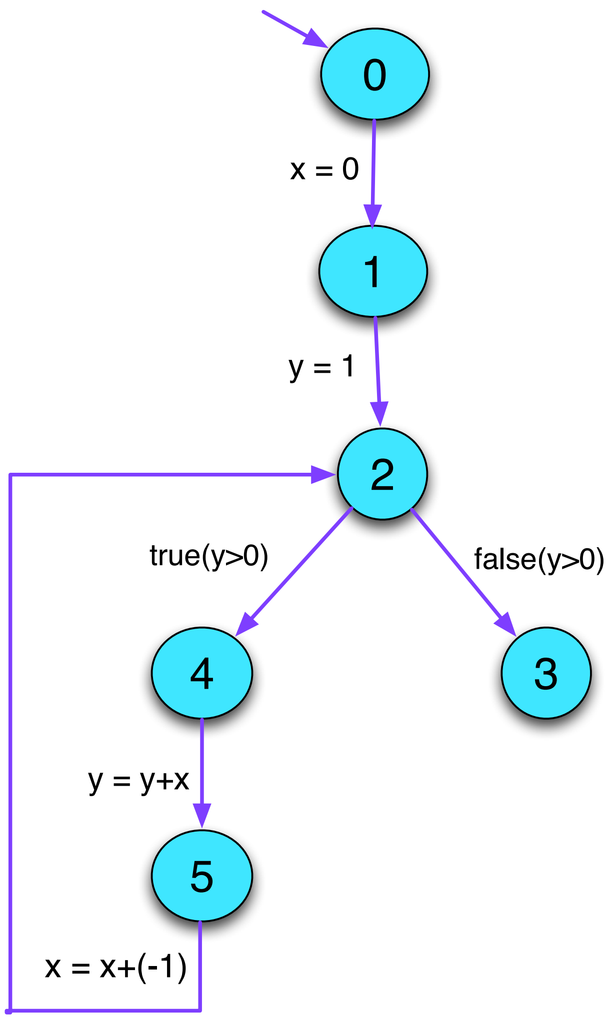
$D_1\, x \supseteq D_2\, x \quad (x \in Vars)$

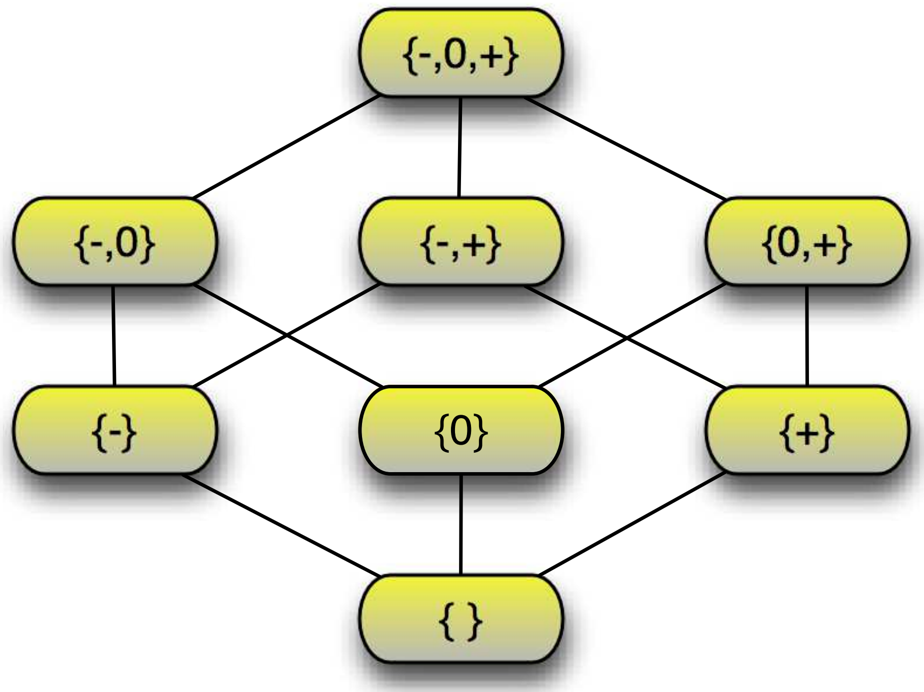
Intuition?

D_1 is at least as precise as D_2 since D_2 admits at least as many signs as D_1

How did we analyze the program?

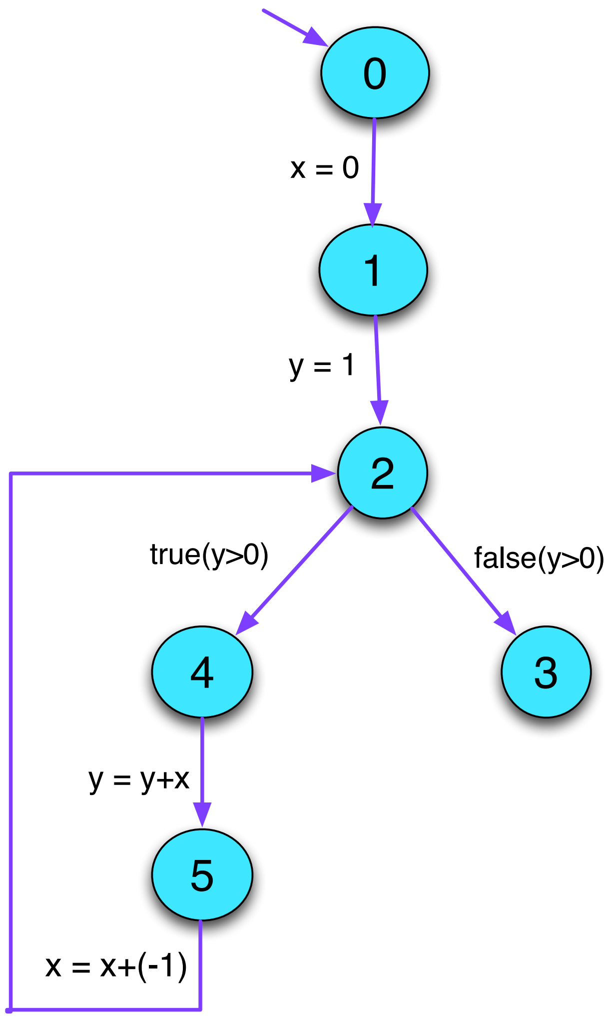


In particular, how did we walk the lattice for y at program point 5?



How is a solution found?

Iterating until a fixed-point is reached



0		1		2		3		4		5	
x	y	x	y	x	y	x	y	x	y	x	y

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- We want to determine the signs of the values of expressions.
- For some sub-expressions, the analysis may yield $\{+, -, 0\}$, which means, it couldn't find out.
- We replace the concrete operators $\Box$ working on values by abstract operators $\Box^\sharp$ working on signs:
- The abstract operators allow to define an abstract evaluation of expressions:

$$[\![e]\!]^\sharp \; : \; (Vars \to Signs) \to Signs$$

Determining the sign of expressions in a Sign-environment is defined by the function $[\![\ ]\!] : Exp \times SignEnv \rightarrow Val$

$$
\begin{array}{lcl}
[\![c]\!]^{\sharp}\, D & = & \begin{cases} \{+\} & \text{if c > 0} \\ \{-\} & \text{if c < 0} \\ \{0\} & \text{if c = 0} \end{cases} \\
[\![v]\!]^{\sharp} & = & D(v) \\
[\![e_1 \,\Box\, e_2]\!]^{\sharp}\, D & = & [\![e_1]\!]^{\sharp}\, D \,\Box^{\sharp}\, [\![e_2]\!]^{\sharp}\, D \\
[\![\Box e]\!]^{\sharp}\, D & = & \Box^{\sharp} [\![e]\!]^{\sharp}\, D
\end{array}
$$

A remark about the notation:
$[\![\ ]\!]$ is given in a ”distributed” form; its first argument appears between the brackets, the second follows the brackets.

Abstract operators working on signs (Addition)

$+^{\#}$	{0}	{+}	{-}	{-, 0}	{-, +}	{0, +}	{-, 0, +}
{0}	{0}	{+}					
{+}							
{-}							
{-, 0}							
{-, +}							
{0, +}							
{-, 0, +}	{-, 0, +}						

Abstract operators working on signs (Multiplication)

$\times^{\#}$	$\{0\}$	$\{+\}$	$\{-\}$	$\cdots$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	
$\{+\}$	$\{0\}$	$\{+\}$	$\{-\}$	
$\{-\}$	$\{0\}$	$\{-\}$	$\{+\}$	
$\{-, 0\}$	$\{0\}$	$\{-, 0\}$	$\{0, +\}$	
$\{-, +\}$	$\{0\}$	$\{-, +\}$	$\{-, +\}$	
$\{0, +\}$	$\{0\}$	$\{0, +\}$	$\{-, 0\}$	
$\{-, 0, +\}$	$\{0\}$	$\{-, 0, +\}$	$\{-, 0, +\}$	

Abstract operators working on signs (unary minus)

$-^{\#}$	{0}	{+}	{-}	{-, 0}	{-, +}	{0, +}	{-, 0, +}
	{0}	{-}	{+}	{+, 0}	{-, +}	{0, -}	{-, 0, +}

Working an example: $D = \{x \mapsto \{+\}, y \mapsto \{+\}\}$

$$
\begin{array}{lcl}
[\![x + 7]\!]^\sharp\, D & = & [\![x]\!]^\sharp\, D \;+^\sharp\; [\![7]\!]^\sharp\, D \\
 & = & \{+\} \;+^\sharp\; \{+\} \\
 & = & \{+\} \\[1em]
[\![x + (-y)]\!]^\sharp\, D & = & \{+\} \;+^\sharp\; (-^\sharp [\![y]\!]^\sharp\, D\,) \\
 & = & \{+\} \;+^\sharp\; (-^\sharp \{+\}) \\
 & = & \{+\} \;+^\sharp\; \{-\} \\
 & = & \{+, -, 0\}
\end{array}
$$

$[\![lab]\!]^\sharp$ is the abstract edge effects associated with edge k.

It depends only on the label *lab*:

$$
\begin{array}{lll}
[\![;]\!]^\sharp\, D & = & D \\
[\![\text{true}\,(e)]\!]^\sharp\, D & = & D \\
[\![\text{false}\,(e)]\!]^\sharp\, D & = & D \\
[\![x = e;]\!]^\sharp\, D & = & D \oplus \{x \mapsto [\![e]\!]^\sharp\, D\} \\
[\![x = M[e];]\!]^\sharp\, D & = & D \oplus \{x \mapsto \{+, -, 0\}\} \\
[\![M[e_1] = e_2;]\!]^\sharp\, D & = & D
\end{array}
$$

... whenever $\qquad D \neq \bot$

These edge effects can be composed to the effect of a path $\pi = k_1 \ldots k_r$:

$$[\![\pi]\!]^\sharp = [\![k_r]\!]^\sharp \circ \ldots \circ [\![k_1]\!]^\sharp$$

Consider a program node v:

- $\rightarrow$ For every path π from program entry *start* to v the analysis should determine for each program variable x the set of all signs that the values of x may have at v as a result of executing π.
- $\rightarrow$ Initially at program start, no information about signs is available.
- $\rightarrow$ The analysis computes a superset of the set of signs as safe information.

$\Longrightarrow$ For each node v, we need the set:

$$\mathcal{S}[v] \quad = \quad \bigcup\{[\![\pi]\!]^\sharp \top \mid \pi : \mathit{start} \rightarrow^* v\}$$

where $\top$ is the function bindig all variables to $\{-, 0, +\}$.

This function describes that we don't know the sign of any variable at program entry.

Question:

How do we compute $\mathcal{S}[u]$ for every program point u?

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Collect all constraints on the values of $\mathcal{S}[u]$ into a system of constraints:

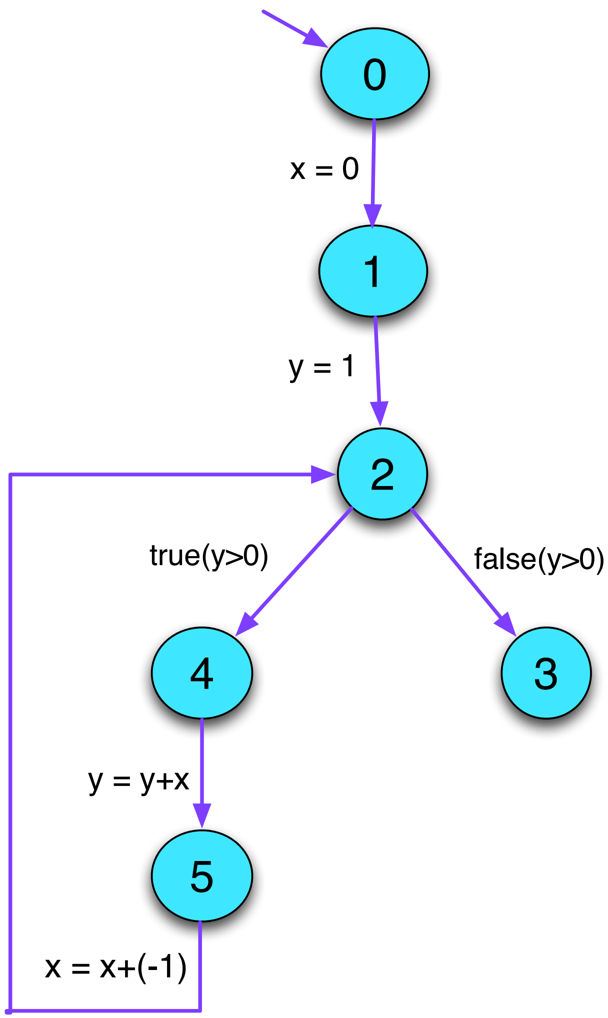
$$\begin{array}{lll} \mathcal{S}[start] & \supseteq & \top \\ \mathcal{S}[v] & \supseteq & [\![k]\!]^\sharp\,(\mathcal{S}[u]) \qquad\qquad k = (u, \_, v) \quad \text{edge} \end{array}$$

Why $\supseteq$?

Wanted:

- a least solution (why least?)
- an algorithm that computes this solution

Example:



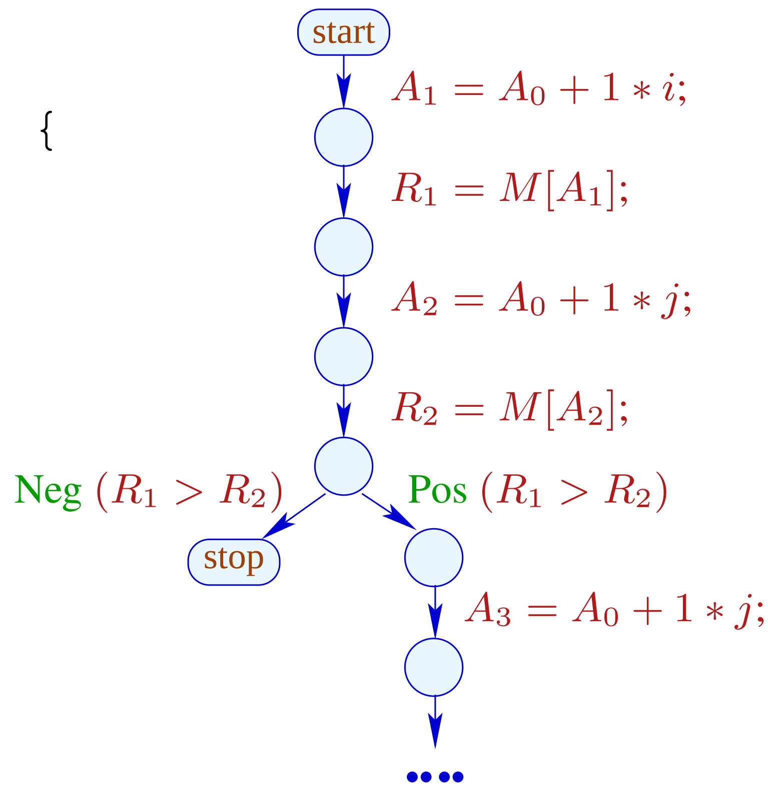
$$\begin{array}{lcl}
\mathcal{S}[0] & \supseteq & \top \\
\mathcal{S}[1] & \supseteq & \mathcal{S}[0] \oplus \{x \mapsto \{0\}\} \\
\mathcal{S}[2] & \supseteq & \mathcal{S}[1] \oplus \{y \mapsto \{+\}\} \\
\mathcal{S}[2] & \supseteq & \mathcal{S}[5] \oplus \{x \mapsto [\![x + (-1)]\!]^\sharp\, \mathcal{S}[5]\} \\
\mathcal{S}[3] & \supseteq & \mathcal{S}[2] \\
\mathcal{S}[4] & \supseteq & \mathcal{S}[2] \\
\mathcal{S}[5] & \supseteq & \mathcal{S}[4] \oplus \{y \mapsto [\![y + x]\!]^\sharp\, \mathcal{S}[4]\}
\end{array}$$

3 An Operational Semantics

Programs are represented as control-flow graphs.

Example:

```
void swap (int i, int j) {
    int t;
    if (a[i] > a[j]) {
        t = a[j];
        a[j] = a[i];
        a[i] = t;
    }
}
```



Thereby, represent:

vertex	program point
start	program start
stop	program exit
edge	labeled with a statement or a condition

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vertex	program point
start	program start
stop	program exit
edge	step of computation

Edge Labelings:

Test : Pos (e) or Neg (e) (better true(e) or false(e))

Assignment : $R = e;$

Load : $R = M[e];$

Store : $M[e_1] = e_2;$

Nop : ;

Execution of a path is a computation.

A computation transforms a state $s = (\rho, \mu)$ where:

$\rho : \mathit{Vars} \to \textbf{int}$	values of variables (contents of symbolic registers)
$\mu : \mathbb{N} \to \textbf{int}$	contents of memory

Every edge $k = (u, lab, v)$ defines a partial transformation

$$[\![k]\!] = [\![lab]\!]$$

of the state:

$$
\begin{aligned}
[\![;]\!]\,(\rho,\mu) &= (\rho,\mu) \\
\\
[\![\mathrm{true}\,(e)]\!]\,(\rho,\mu) &= (\rho,\mu) && \text{if } [\![e]\!]\,\rho \neq 0 \\
[\![\mathrm{false}\,(e)]\!]\,(\rho,\mu) &= (\rho,\mu) && \text{if } [\![e]\!]\,\rho = 0
\end{aligned}
$$

$$[\![;]\!]\,(\rho, \mu) \quad = \quad (\rho, \mu)$$

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// $[\![e]\!]$: evaluation of the expression e, e.g.

// $[\![x + y]\!]\,\{x \mapsto 7, y \mapsto -1\} = 6$

// $[\![!(x == 4)]\!]\,\{x \mapsto 5\} = 1$

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$$[\![R = e;]\!]\,(\rho,\mu) = (\,\rho \oplus \{R \mapsto [\![e]\!]\,\rho\}\,,\mu)$$

// where "$\oplus$" modifies a mapping at a given argument

$$[\![R = M[e];]\!]\,(\rho, \mu) \quad = \quad (\boxed{\rho \oplus \{R \mapsto \mu([\![e]\!]\,\rho))\}}, \mu)$$

$$[\![M[e_1] = e_2;]\!]\,(\rho, \mu) \quad = \quad (\rho, \boxed{\mu \oplus \{[\![e_1]\!]\,\rho \mapsto [\![e_2]\!]\,\rho\}})$$

Example:

$[\![x = x + 1;]\!]\,(\{x \mapsto 5\}, \mu) = (\rho, \mu)$ where

$$\begin{aligned} \rho &= \{x \mapsto 5\} \oplus \{x \mapsto [\![x + 1]\!]\,\{x \mapsto 5\}\} \\ &= \{x \mapsto 5\} \oplus \{x \mapsto 6\} \\ &= \{x \mapsto 6\} \end{aligned}$$

A path $\pi = k_1 k_2 \dots k_m$ defines a computation in the state s if

$$s \in \mathit{def}\,([\![k_m]\!] \circ \ldots \circ [\![k_1]\!])$$

The result of the computation is $[\![\pi]\!]\,s = ([\![k_m]\!] \circ \ldots \circ [\![k_1]\!])\,s$

The approach:

A static analysis needs to collect correct and hopefully precise information about a program in a terminating computation.

Concepts:

- partial orders relate information for their contents/quality/precision,
- least upper bounds combine information in the best possible way,
- monotonic functions preserve the order, prevent loss of collected information, prevent oscillation.

4 Complete Lattices

A set $\mathbb{D}$ together with a relation $\sqsubseteq \subseteq \mathbb{D} \times \mathbb{D}$ is a partial order if for all $a, b, c \in \mathbb{D}$,

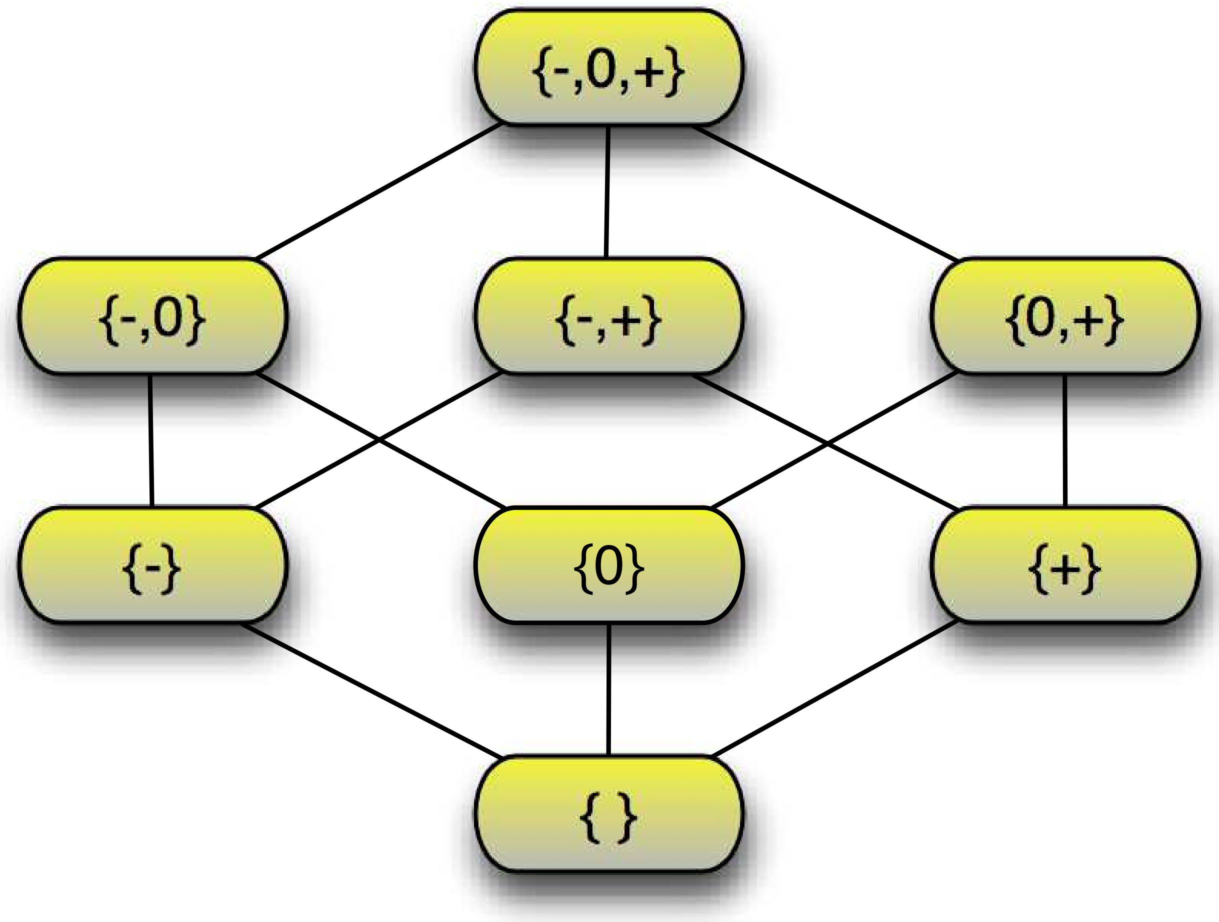
$$
\begin{array}{lll}
a \sqsubseteq a & & \textit{reflexivity} \\
a \sqsubseteq b \wedge b \sqsubseteq a & \Longrightarrow \quad a = b & \textit{anti-symmetry} \\
a \sqsubseteq b \wedge b \sqsubseteq c & \Longrightarrow \quad a \sqsubseteq c & \textit{transitivity}
\end{array}
$$

Intuition: $\sqsubseteq$ represents precision.

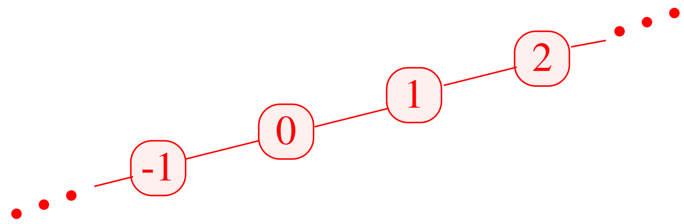
By convention: $a \sqsubseteq b$ means a is at least as precise as b.

Examples:

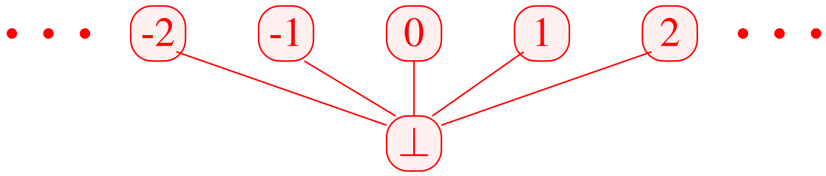
1. The rules-of-sign analysis uses the following lattice $\mathbb{D} = 2^{\{-,0,+\}}$ with the relation "$\subseteq$" :



2. $\mathbb{Z}$ with the relation "$\leq$" :



3. $\mathbb{Z}_\perp = \mathbb{Z} \cup \{\perp\}$ with the ordering:



$d \in \mathbb{D}$ is called upper bound for $X \subseteq \mathbb{D}$ if

$$x \sqsubseteq d \qquad \text{for all } x \in X$$

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1. d is an upper bound and
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The least upper bound is the youngest common ancestor in the pictorial representation of lattices.

Intuition: It is the best combined information for X.

Caveat:

- $\{0, 2, 4, \ldots\} \subseteq \mathbb{Z}$ has no upper bound!
- $\{0, 2, 4\} \subseteq \mathbb{Z}$ has the upper bounds $4, 5, 6, \ldots$

A partially ordered set $\mathbb{D}$ is a complete lattice (cl) if every subset $X \subseteq \mathbb{D}$ has a least upper bound $\bigsqcup X \in \mathbb{D}$.

Note:

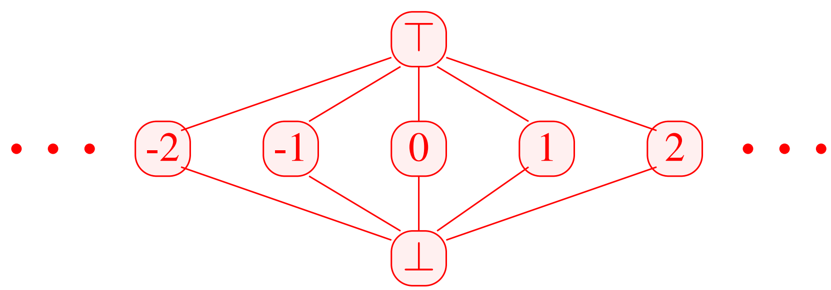
Every complete lattice has

$\rightarrow$ a least element $\bot = \bigsqcup \emptyset \in \mathbb{D}$;

$\rightarrow$ a greatest element $\top = \bigsqcup \mathbb{D} \in \mathbb{D}$.

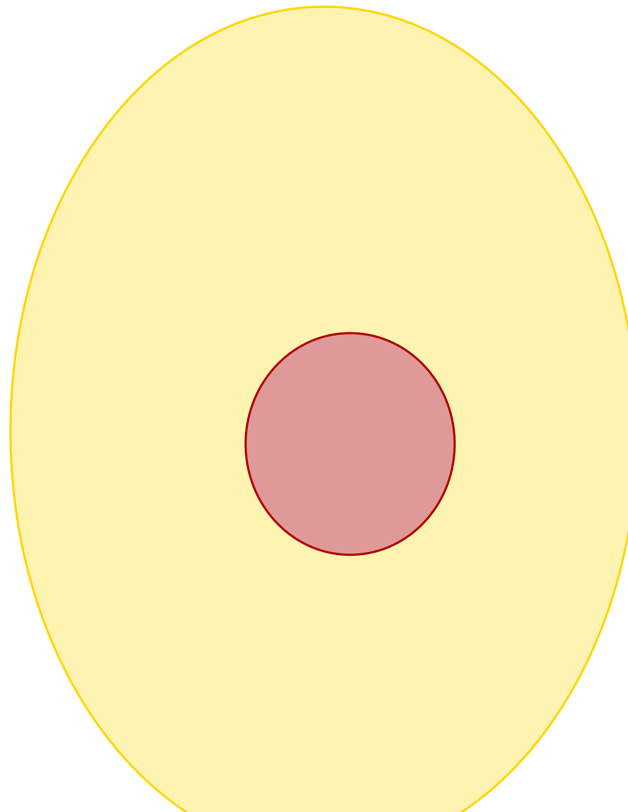
Examples:

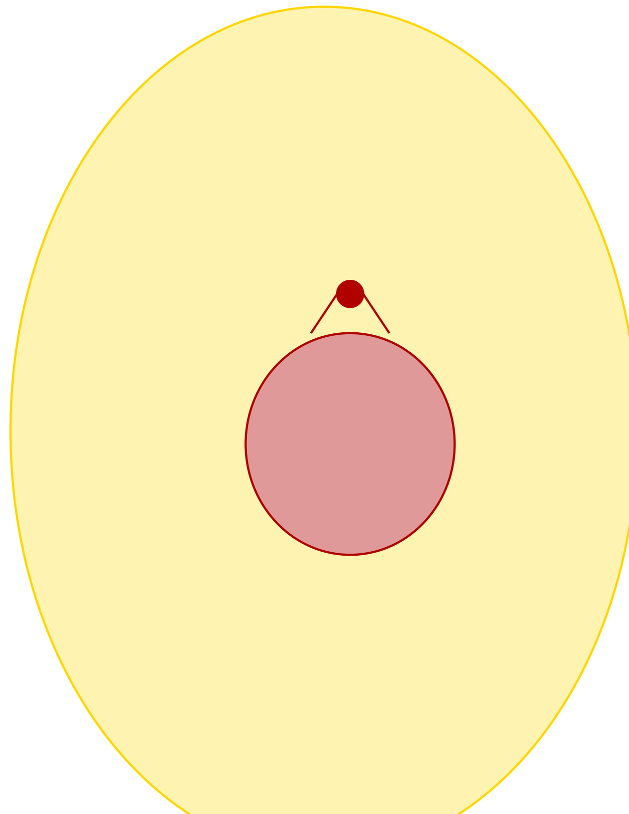
1. $\mathbb{D} = 2^{\{-,0,+\}}$ with $\sqsubseteq$ is a complete lattice
2. $\mathbb{D} = \mathbb{Z}$ with "$\leq$" is not a complete lattice.
3. $\mathbb{D} = \mathbb{Z}_\bot$ is also not a complete lattice
4. With an extra element $\top$, we obtain the flat lattice $\mathbb{Z}_\bot^\top = \mathbb{Z} \cup \{\bot, \top\}$:

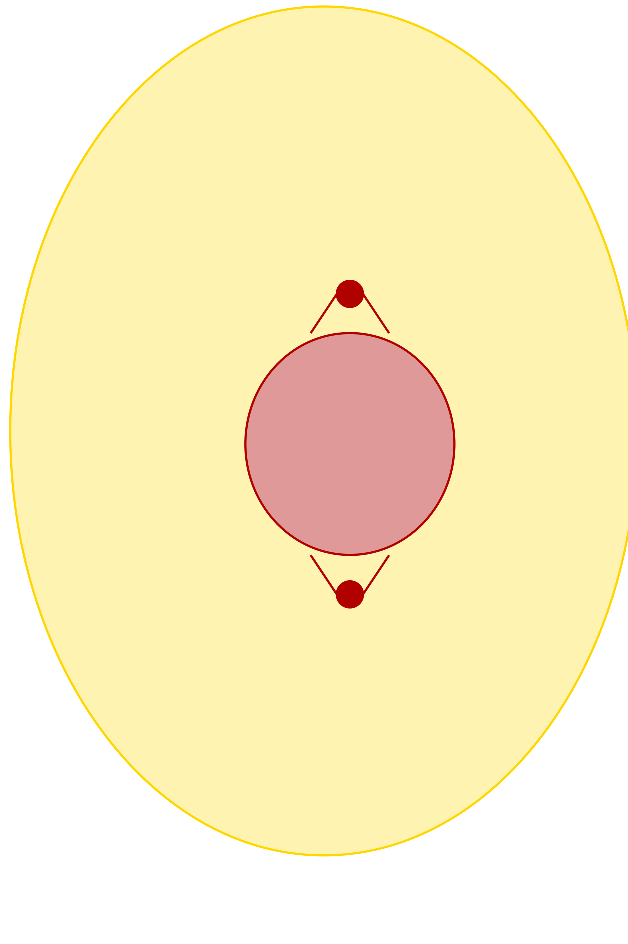


Theorem:

If $\mathbb{D}$ is a complete lattice, then every subset $X \subseteq \mathbb{D}$ has a greatest lower bound $\bigsqcap X$.







Back to the system of constraints for Rules-of-Signs Analysis!

$$\begin{array}{lll} \mathcal{S}[start] & \sqsupseteq & \top \\ \mathcal{S}[v] & \sqsupseteq & [\![k]\!]^\sharp\,(\mathcal{S}[u]) \qquad\qquad k = (u, \_, v) \quad \text{edge} \end{array}$$

Combine all constraints for each variable v by applying the least-upper-bound operator $\bigsqcup$:

$$\mathcal{S}[v] \quad \sqsupseteq \quad \bigsqcup\{[\![k]\!]^\sharp\,(\mathcal{S}[u]) \mid k = (u, \_, v) \ \text{ edge}\}$$

Correct because:

$$x \sqsupseteq d_1 \wedge \ldots \wedge x \sqsupseteq d_k \quad \text{iff} \quad x \sqsupseteq \bigsqcup\{d_1, \ldots, d_k\}$$

Our generic form of the systems of constraints:

$$x_i \quad \sqsupseteq \quad f_i(x_1, \ldots, x_n) \qquad\qquad (*)$$

Relation to the running example:

x_i	unknown	here:	$\mathcal{S}[u]$
$\mathbb{D}$	values	here:	$Signs$
$\sqsubseteq \quad \subseteq \quad \mathbb{D} \times \mathbb{D}$	ordering relation	here:	$\subseteq$
f_i: $\mathbb{D}^n \to \mathbb{D}$	constraint	here:	...

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(2) $\mathbb{D}_1 = \mathbb{D}_2 = \mathbb{Z}$ (with the ordering "$\leq$"). Then:

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- $\mathsf{dec}\,x = x - 1$ is monotonic.

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- $\mathsf{inc}\,x = x + 1$ is monotonic.
- $\mathsf{dec}\,x = x - 1$ is monotonic.
- $\mathsf{inv}\,x = -x$ is not monotonic

Theorem:

If $f_1 : \mathbb{D}_1 \to \mathbb{D}_2$ and $f_2 : \mathbb{D}_2 \to \mathbb{D}_3$ are monotonic, then also $f_2 \circ f_1 : \mathbb{D}_1 \to \mathbb{D}_3$

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Wanted: least solution for:

$$x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n \qquad (*)$$

where all $f_i : \mathbb{D}^n \to \mathbb{D}$ are monotonic.

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Idea:

- Consider $F : \mathbb{D}^n \to \mathbb{D}^n$ where

$$F(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \quad \text{with} \quad y_i = f_i(x_1, \ldots, x_n).$$

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- If all f_i are monotonic, then also F
- We successively approximate a solution from below. We construct:

  $$\underline{\perp}, \quad F\,\underline{\perp}, \quad F^2\,\underline{\perp}, \quad F^3\,\underline{\perp}, \quad \ldots$$

Intuition: This iteration eliminates unjustified assumptions.

Hope: We eventually reach a solution!

Example: $\mathbb{D} = 2^{\{a,b,c\}}, \quad \sqsubseteq \; = \; \subseteq$

$$
\begin{array}{lll}
x_1 & \supseteq & \{a\} \cup x_3 \\
x_2 & \supseteq & x_3 \cap \{a, b\} \\
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\end{array}
$$

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The Iteration:

	0	1	2	3	4
x_1	$\emptyset$				
x_2	$\emptyset$				
x_3	$\emptyset$				

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x_1	$\emptyset$	$\{a\}$	$\{a,c\}$	$\{a,c\}$	dito
x_2	$\emptyset$	$\emptyset$	$\emptyset$	$\{a\}$	
x_3	$\emptyset$	$\{c\}$	$\{a,c\}$	$\{a,c\}$	

Theorem

- $\underline{\perp}, F\,\underline{\perp}, F^2\,\underline{\perp}, \dots$ form an ascending chain :

$$\underline{\perp} \quad \sqsubseteq \quad F\,\underline{\perp} \quad \sqsubseteq \quad F^2\,\underline{\perp} \quad \sqsubseteq \quad \dots$$

- If $F^k\,\underline{\perp} = F^{k+1}\,\underline{\perp}$, F^k is the least solution.
- If all ascending chains are finite, such a k always exists.

Theorem

- $\underline{\bot}, F\,\underline{\bot}, F^2\,\underline{\bot}, \ldots$ form an ascending chain :

$$\underline{\bot} \quad \sqsubseteq \quad F\,\underline{\bot} \quad \sqsubseteq \quad F^2\,\underline{\bot} \quad \sqsubseteq \quad \ldots$$

- If $F^k\,\underline{\bot} = F^{k+1}\,\underline{\bot}$, a solution is obtained, which is the least one.
- If all ascending chains are finite, such a k always exists.

If $\mathbb{D}$ is finite, a solution can be found that is definitely the least solution.

Question: What, if $\mathbb{D}$ is not finite?

Theorem Knaster – Tarski

Assume $\mathbb{D}$ is a complete lattice. Then every monotonic function $f : \mathbb{D} \to \mathbb{D}$ has a least fixed point $d_0 \in \mathbb{D}$.

Application:

Assume $$x_i \sqsupseteq f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n \qquad (*)$$

is a system of constraints where all $f_i : \mathbb{D}^n \to \mathbb{D}$ are monotonic.

$\Longrightarrow$ least solution of $(*)$ $==$ least fixed point of F

Example 1: $\mathbb{D} = 2^U, \quad f\,x = x \cap a \cup b$

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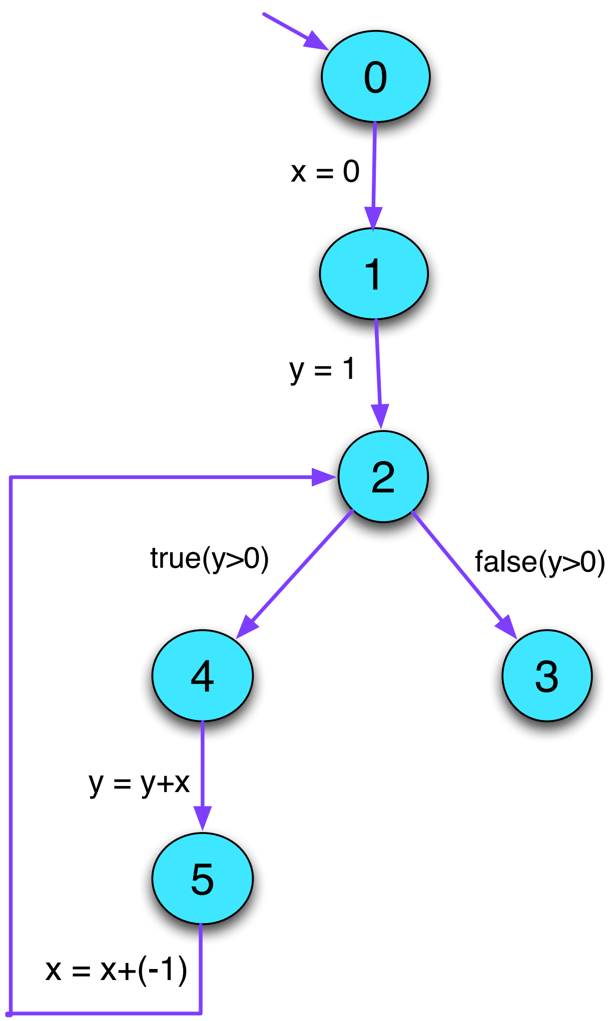
f	$f^k\,\bot$	$f^k\,\top$
0	$\emptyset$	U
1	b	$a \cup b$
2	b	$a \cup b$

Conclusion:

Systems of inequalities can be solved through fixed-point iteration, i.e., by repeated evaluation of right-hand sides

Caveat: Naive fixed-point iteration is rather inefficient

Example:

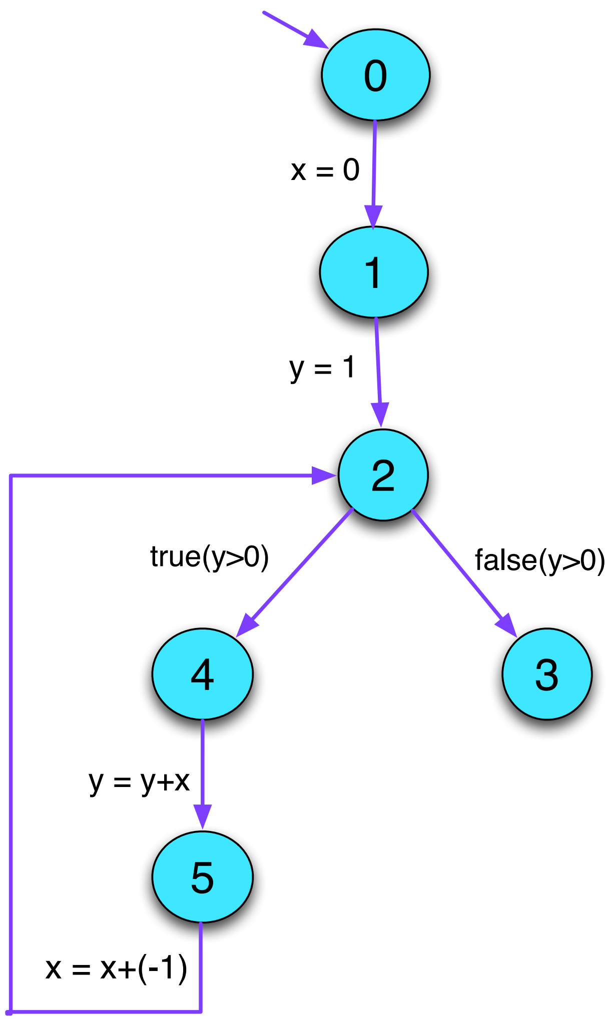


0		1		2		3		4		5	
x	y	x	y	x	y	x	y	x	y	x	y

Idea: Round Robin Iteration

Instead of accessing the values of the last iteration, always use the current values of unknowns

Example:



0		1		2		3		4		5	
x	y	x	y	x	y	x	y	x	y	x	y

The code for Round Robin Iteration in Java looks as follows:

```
for (i = 1; i ≤ n; i++) x_i = ⊥;
do {
        finished = true;
        for (i = 1; i ≤ n; i++) {
                new = f_i(x_1, ..., x_n);
                if (!(x_i ⊒ new)) {
                        finished = false;
                        x_i = x_i ⊔ new;
                }
        }
} while (!finished);
```

What we have learned:

- The information derived by static program analysis is partially ordered in a complete lattice.
- the partial order represents information content/precision of the lattice elements.
- least upper-bound combines information in the best possible way.
- Monotone functions prevent loss of information.

For a complete lattice $\mathbb{D}$, consider systems:

$$\begin{array}{lll} \mathcal{I}[start] & \sqsupseteq & d_0 \\ \mathcal{I}[v] & \sqsupseteq & [\![k]\!]^\sharp\,(\mathcal{I}[u]) \qquad k = (u, \_, v) \quad \text{edge} \end{array}$$

where $d_0 \in \mathbb{D}$ and all $[\![k]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ are monotonic ...

Wanted: MOP (Merge Over all Paths)

$$\mathcal{I}^*[v] = \bigsqcup \{ [\![\pi]\!]^\sharp\, d_0 \mid \pi : start \to^* v \}$$

Theorem Kam, Ullman 1975

Assume $\mathcal{I}$ is a solution of the constraint system. Then:

$$\mathcal{I}[v] \sqsupseteq \mathcal{I}^*[v] \qquad \text{for every} \quad v$$

In particular: $\mathcal{I}[v] \sqsupseteq [\![\pi]\!]^\sharp\, d_0$ for every $\pi : start \to^* v$

Disappointment: Are solutions of the constraint system just upper bounds?

Answer: In general: yes

Notable exception, all functions $[\![k]\!]^\sharp$ are distributive.

The function $f : \mathbb{D}_1 \to \mathbb{D}_2$ is called distributive, if
$f\,(\bigsqcup X) = \bigsqcup\{f\,x \mid x \in X\}$ for all $\emptyset \neq X \subseteq \mathbb{D}$;

Remark: If $f : \mathbb{D}_1 \to \mathbb{D}_2$ is distributive, then it is also monotonic

Theorem Kildall 1972

Assume all v are reachable from *start*.

Then: If all effects of edges $[\![k]\!]^\sharp$ are distributive, $\mathcal{I}^*[v] = \mathcal{I}[v]$ holds for all v.

Question: Are the edge effects of the Rules-of-Sign analysis distributive?

5 Constant Propagation

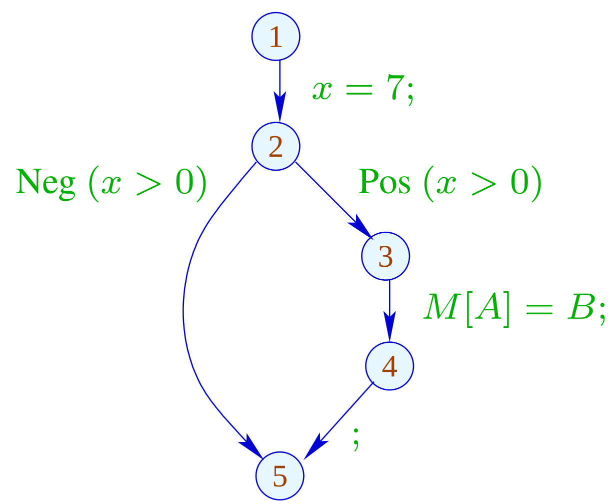
Goal: Execute as much of the code at compile-time as possible!

Example:

$x = 7;$

$\text{if } (x > 0)$

$\quad M[A] = B;$



Obviously, x has always the value 7

Thus, the memory access is always executed

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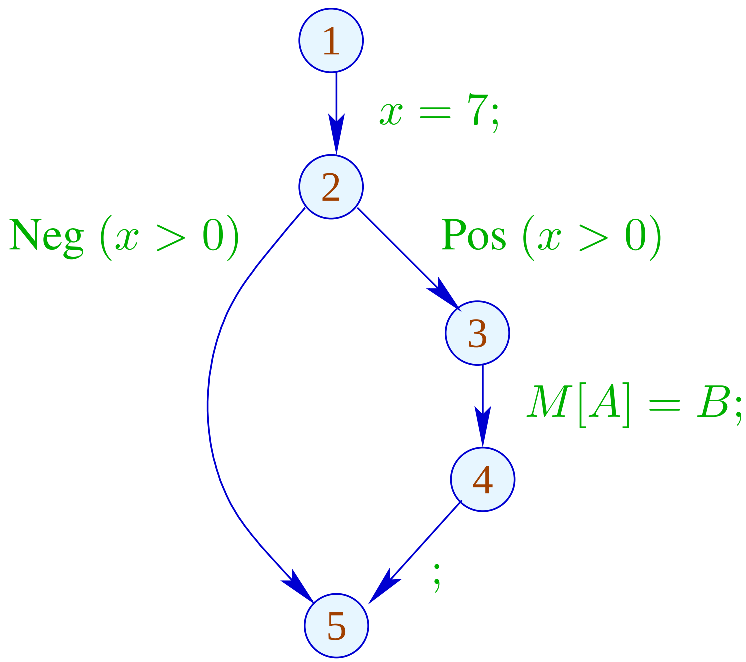

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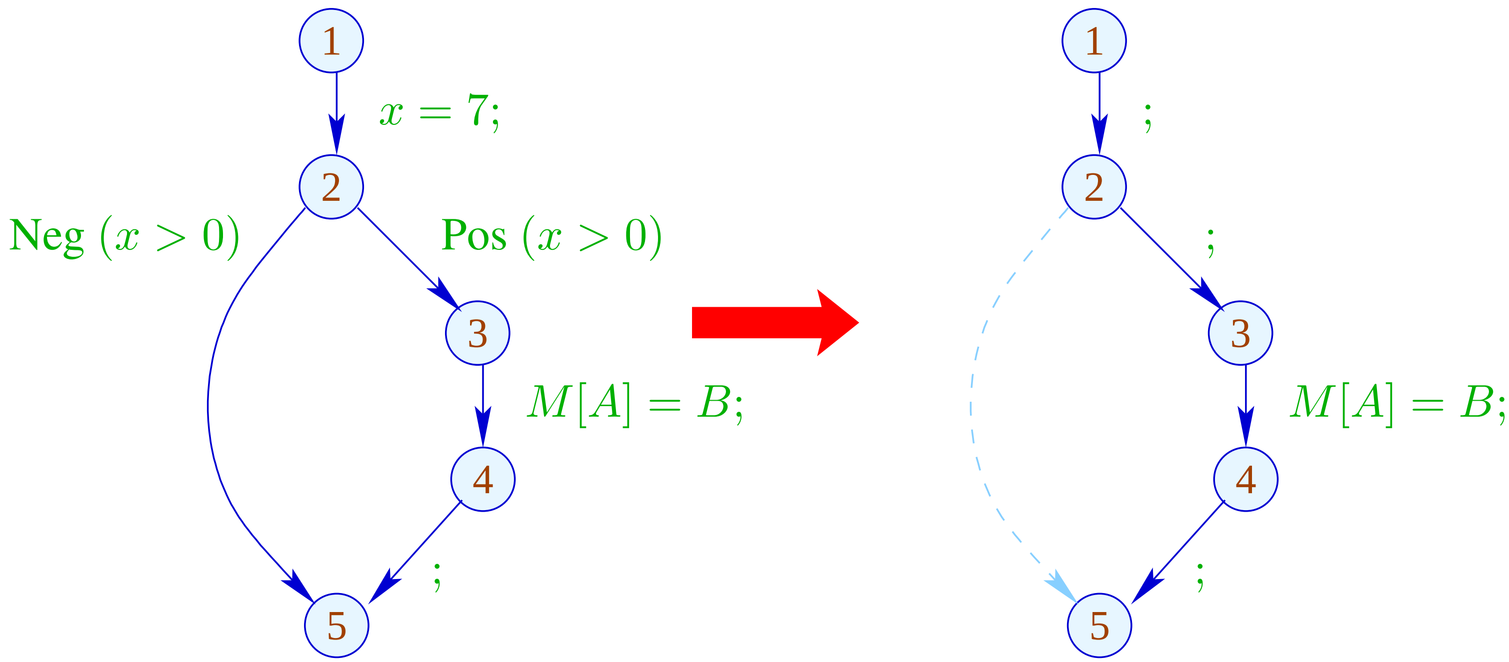

Idea:

Design an analysis that for every program point u determines the values that variables definitely have at u;

As a side effect, it also tells whether u can be reached at all

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As a side effect, it also tells whether u can be reached at all

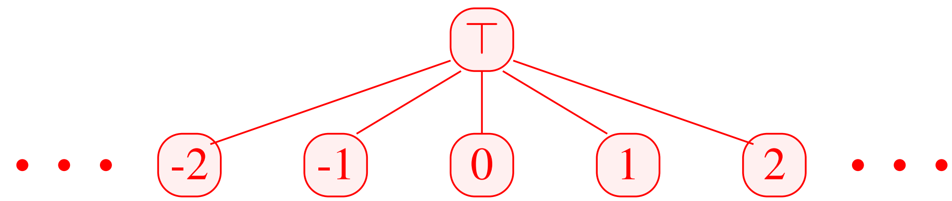
We need to design a complete lattice for this analysis.

It has a nice relation to the operational semantics of our tiny programming language.

As in the case of the Rules-of-Signs analysis the complete lattice is constructed in two steps.

(1) The potential values of variables:

$$\mathbb{Z}^{\top} = \mathbb{Z} \cup \{\top\} \qquad \text{with} \quad x \sqsubseteq y \quad \text{iff } y = \top \text{ or } x = y$$



Caveat: $\mathbb{Z}^\top$ is not a complete lattice in itself

(2) $\mathbb{D} = (\mathit{Vars} \to \mathbb{Z}^\top)_\perp = (\mathit{Vars} \to \mathbb{Z}^\top) \cup \{\perp\}$

// $\perp$ denotes: “not reachable”

with $D_1 \sqsubseteq D_2$ iff $\perp = D_1$ or

$D_1\, x \sqsubseteq D_2\, x \quad (x \in \mathit{Vars})$

Remark: $\mathbb{D}$ is a complete lattice

For every edge $k = (\_, lab, \_)$, construct an effect function $[\![k]\!]^\sharp = [\![lab]\!]^\sharp \;:\; \mathbb{D} \to \mathbb{D}$ which simulates the concrete computation.

Obviously, $[\![lab]\!]^\sharp \bot = \bot$ for all lab

Now let $\bot \neq D \in Vars \to \mathbb{Z}^\top$.

Idea:

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$\Longrightarrow$

We must replace the concrete operators $\Box$ by abstract operators $\Box^\sharp$ which can handle $\top$:

$$a \,\Box^\sharp\, b = \begin{cases} \top & \text{if} \quad a = \top \text{ or } b = \top \\ a \,\Box\, b & \text{otherwise} \end{cases}$$

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- The abstract operators allow to define an abstract evaluation of expressions:

$$[\![e]\!]^\sharp \;:\; (\mathit{Vars} \to \mathbb{Z}^\top) \to \mathbb{Z}^\top$$

Abstract evaluation of expressions is like the concrete evaluation — but with abstract values and operators. Here:

$$
\begin{array}{lll}
[\![c]\!]^\sharp \, D & = & c \\
[\![e_1 \,\square\, e_2]\!]^\sharp \, D & = & [\![e_1]\!]^\sharp \, D \,\square^\sharp\, [\![e_2]\!]^\sharp \, D
\end{array}
$$

... analogously for unary operators

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... analogously for unary operators

Example: $D = \{x \mapsto 2, y \mapsto \top\}$

$$\begin{array}{lll} [\![x + 7]\!]^\sharp\, D & = & [\![x]\!]^\sharp\, D \;+^\sharp\; [\![7]\!]^\sharp\, D \\ & = & 2 \;+^\sharp\; 7 \\ & = & 9 \\ [\![x - y]\!]^\sharp\, D & = & 2 \;-^\sharp\; \top \\ & = & \top \end{array}$$

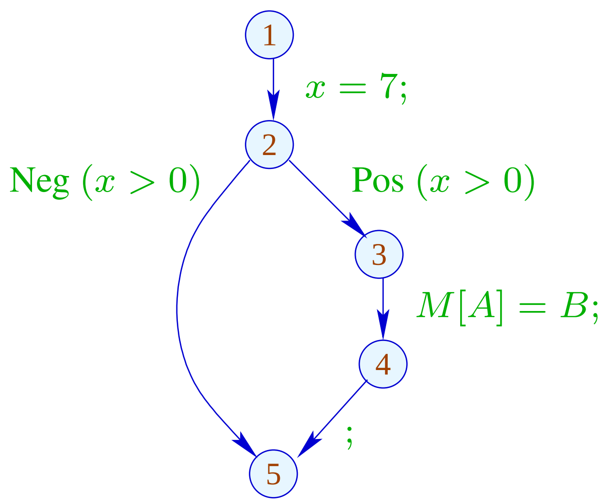
Thus, we obtain the following abstract edge effects $[\![lab]\!]^\sharp$:

$$
\begin{array}{lll}
[\![;]\!]^\sharp\, D & = & D \\
[\![\text{true}\,(e)]\!]^\sharp\, D & = & \begin{cases} \bot & \text{if} \quad 0 = [\![e]\!]^\sharp\, D \quad \text{definitely false} \\ D & \text{otherwise} \qquad\qquad\;\; \text{possibly true} \end{cases} \\
[\![\text{false}\,(e)]\!]^\sharp\, D & = & \begin{cases} D & \text{if} \quad 0 \sqsubseteq [\![e]\!]^\sharp\, D \quad \text{possibly false} \\ \bot & \text{otherwise} \qquad\qquad\;\; \text{definitely true} \end{cases} \\
[\![x = e;]\!]^\sharp\, D & = & D \oplus \{x \mapsto [\![e]\!]^\sharp\, D\} \\
[\![x = M[e];]\!]^\sharp\, D & = & D \oplus \{x \mapsto \top\} \\
[\![M[e_1] = e_2;]\!]^\sharp\, D & = & D
\end{array}
$$

... whenever $D \neq \bot$

At *start*, we have $D_\top = \{x \mapsto \top \mid x \in \mathit{Vars}\}$.

Example:



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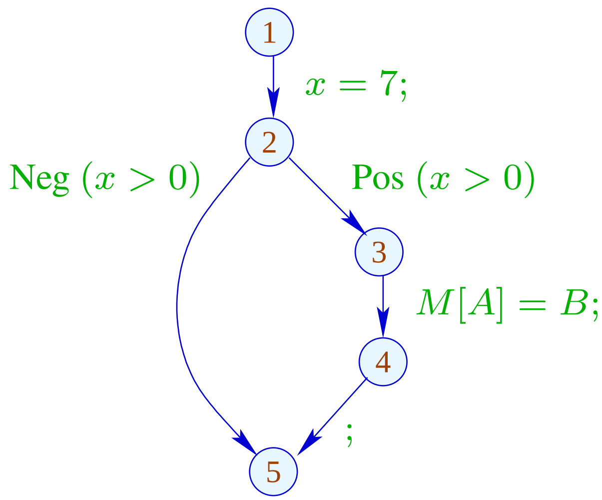

1	$\{x \mapsto \top\}$
2	$\{x \mapsto 7\}$
3	$\{x \mapsto 7\}$
4	$\{x \mapsto 7\}$
5	$\bot \sqcup \{x \mapsto 7\} = \{x \mapsto 7\}$

The abstract effects of edges $[\![k]\!]^\sharp$ are again composed to form the effects of paths $\pi = k_1 \ldots k_r$ by:

$$[\![\pi]\!]^\sharp = [\![k_r]\!]^\sharp \circ \ldots \circ [\![k_1]\!]^\sharp \quad : \mathbb{D} \to \mathbb{D}$$

Idea for Correctness: Abstract Interpretation

Cousot, Cousot 1977

Establish a description relation Δ between the concrete values and their descriptions with:

$$x \,\Delta\, a_1 \quad \wedge \quad a_1 \sqsubseteq a_2 \quad \Longrightarrow \quad x \,\Delta\, a_2$$

Concretization: $\gamma\, a = \{x \mid x \,\Delta\, a\}$

// returns the set of described values

(1) Values: $\Delta \;\subseteq \mathbb{Z} \times \mathbb{Z}^\top$

$$z \,\Delta\, a \quad \text{iff} \quad z = a \vee a = \top$$

Concretization:

$$\gamma\, a = \begin{cases} \{a\} & \text{if} \quad a \sqsubset \top \\ \mathbb{Z} & \text{if} \quad a = \top \end{cases}$$

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(2) Variable Bindings: $\Delta \subseteq (\mathit{Vars} \to \mathbb{Z}) \times (\mathit{Vars} \to \mathbb{Z}^\top)_\perp$

$$\rho \,\Delta\, D \quad \text{iff} \quad D \neq \perp \wedge \rho\, x \sqsubseteq D\, x \quad (x \in \mathit{Vars})$$

Concretization:

$$\gamma\, D = \begin{cases} \emptyset & \text{if} \quad D = \perp \\ \{\rho \mid \forall\, x : (\rho\, x) \,\Delta\, (D\, x)\} & \text{otherwise} \end{cases}$$

Example: $\{x \mapsto 1, y \mapsto -7\} \;\Delta\; \{x \mapsto \top, y \mapsto -7\}$

(3) States:

$$\Delta \;\subseteq\; ((\mathit{Vars} \to \mathbb{Z}) \times (\mathbb{N} \to \mathbb{Z})) \times (\mathit{Vars} \to \mathbb{Z}^\top)_\bot$$

$$(\rho, \mu) \;\Delta\; D \qquad \text{iff} \qquad \rho \;\Delta\; D$$

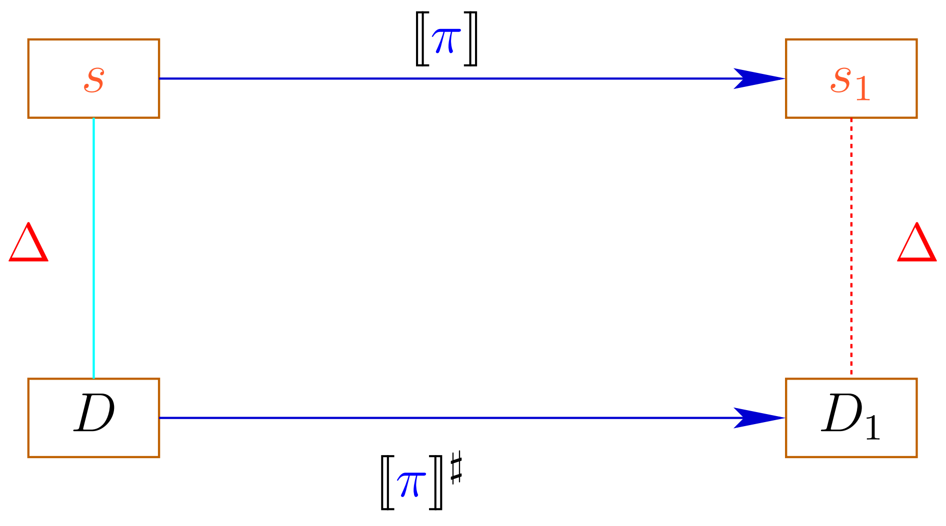
Concretization:

$$\gamma\, D = \begin{cases} \emptyset & \text{if} \quad D = \bot \\ \{(\rho, \mu) \mid \forall\, x : \; (\rho\, x) \;\Delta\; (D\, x)\} & \text{otherwise} \end{cases}$$

We show correctness:

(*) If $s \, \Delta \, D$ and $[\![\pi]\!] \, s$ is defined, then:

$$([\![\pi]\!] \, s) \;\; \Delta \;\; ([\![\pi]\!]^\sharp \, D)$$



The abstract semantics simulates the concrete semantics

In particular:

$$[\![\pi]\!]\, s \in \gamma\left([\![\pi]\!]^\sharp\, D\right)$$

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In practice, this means for example that $D\, x = -7$ implies:

$$\begin{array}{lrcl} & \rho'\, x & = & -7 \quad \text{for all} \quad \rho' \in \gamma\, D \\ \Longrightarrow & \rho_1\, x & = & -7 \quad \text{for} \quad (\rho_1, \_) = [\![\pi]\!]\, s \end{array}$$

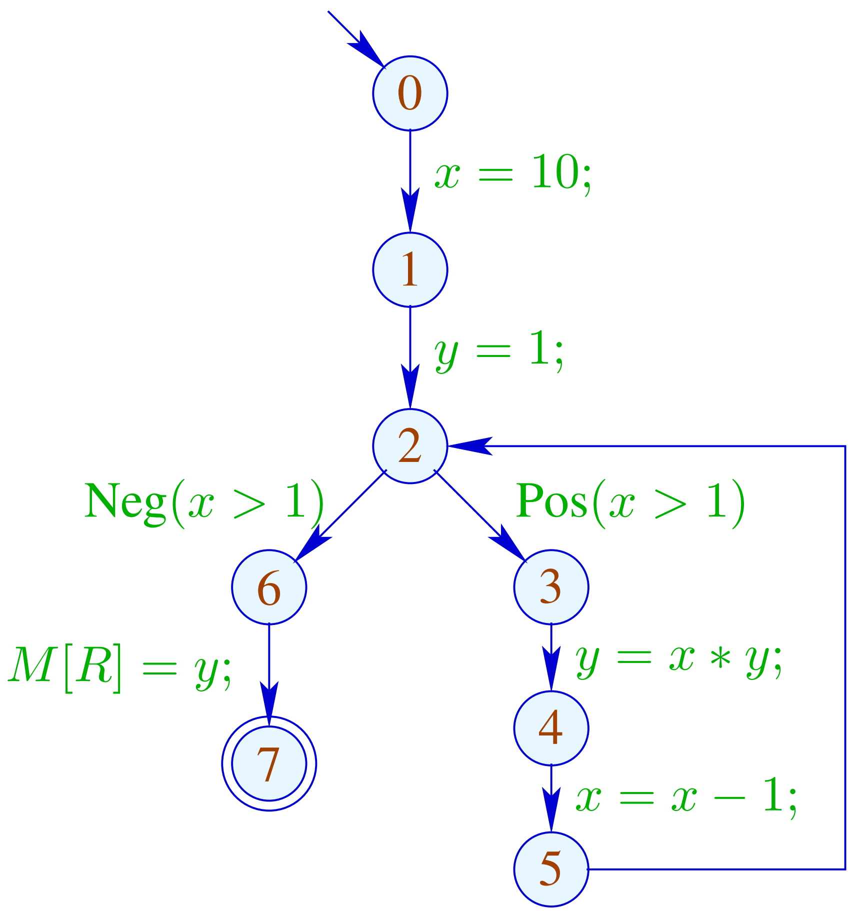
The MOP-Solution:

$$\mathcal{D}^*[v] \;=\; \bigsqcup\{[\![\pi]\!]^\sharp\, D_\top \mid \pi : start \to^* v\}$$

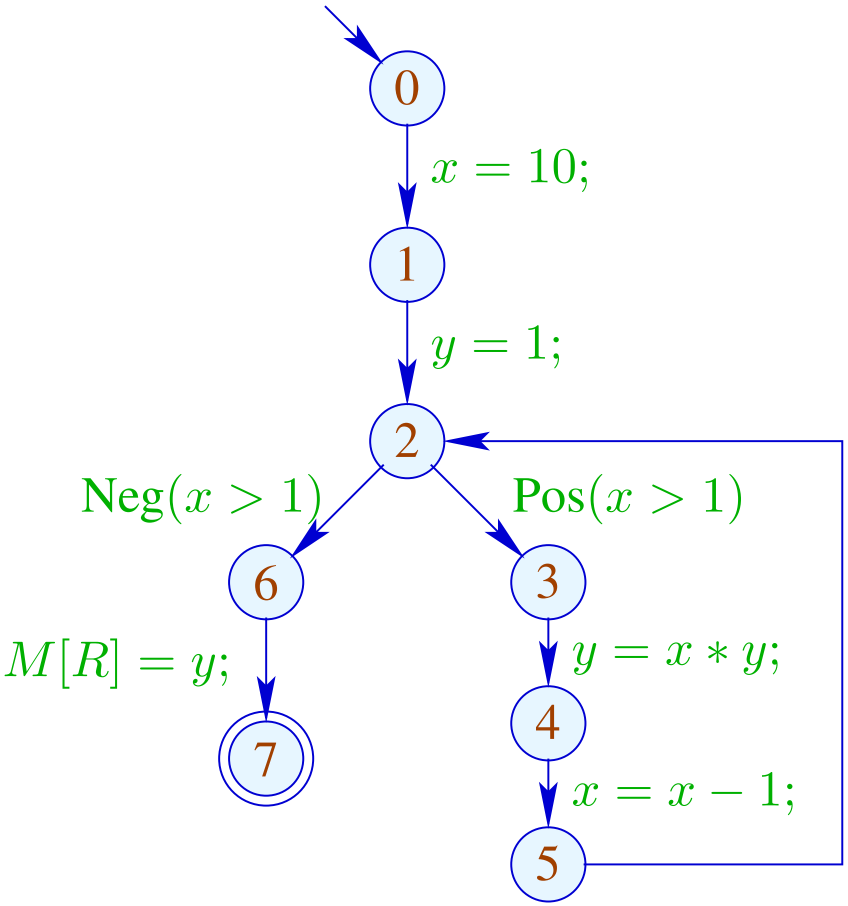
where $\quad D_\top\, x = \top \qquad (x \in Vars)$.

In order to approximate the MOP, we use our constraint system

Example:

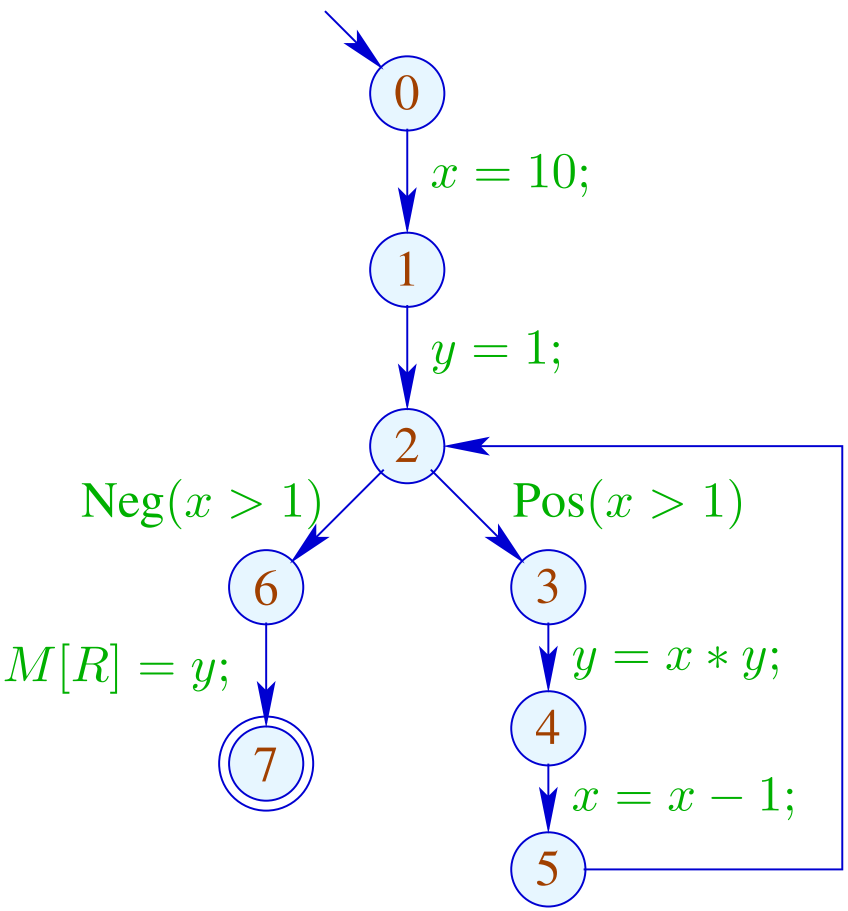


Example:



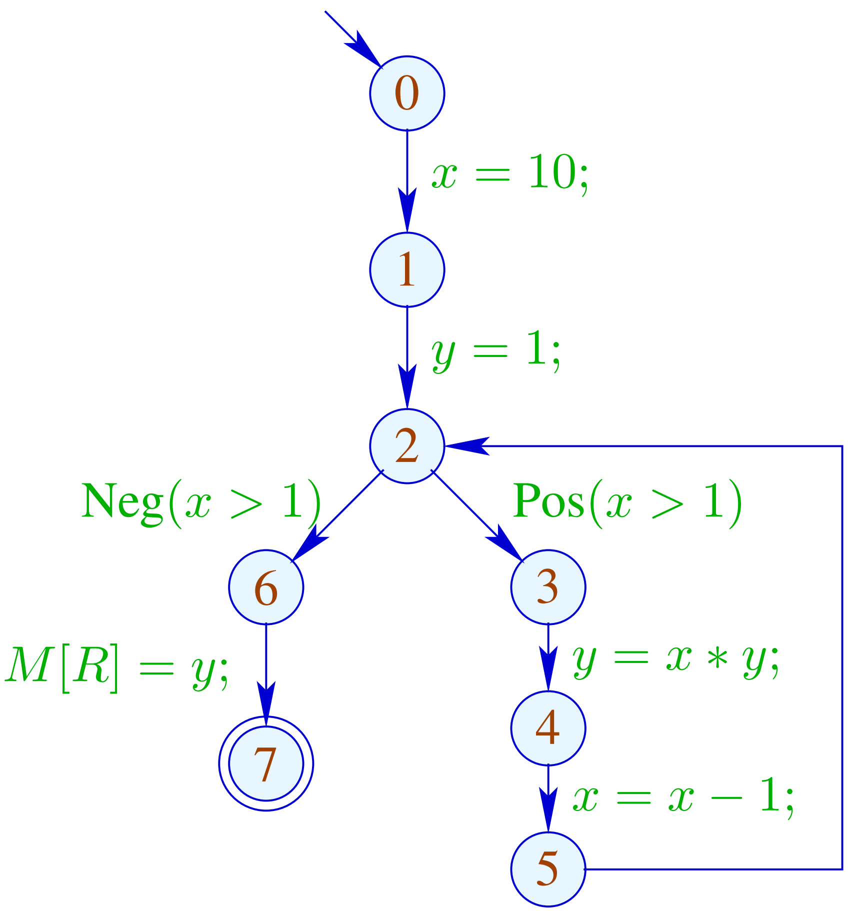
	1	
	x	y
0	$\top$	$\top$
1	10	$\top$
2	10	1
3	10	1
4	10	10
5	9	10
6	$\bot$	
7	$\bot$	

Example:



	1		2	
	x	y	x	y
0	$\top$	$\top$	$\top$	$\top$
1	10	$\top$	10	$\top$
2	10	1	$\top$	$\top$
3	10	1	$\top$	$\top$
4	10	10	$\top$	$\top$
5	9	10	$\top$	$\top$
6	$\perp$		$\top$	$\top$
7	$\perp$		$\top$	$\top$

Example:



	1		2		3	
	x	y	x	y	x	y
0	$\top$	$\top$	$\top$	$\top$	dito	
1	10	$\top$	10	$\top$		
2	10	1	$\top$	$\top$		
3	10	1	$\top$	$\top$		
4	10	10	$\top$	$\top$		
5	9	10	$\top$	$\top$		
6	$\bot$		$\top$	$\top$		
7	$\bot$		$\top$	$\top$		

Concrete vs. Abstract Execution:

Although program and all initial values are given, abstract execution does not compute the result!

On the other hand, fixed-point iteration is guaranteed to terminate:

For n program points and m variables, we maximally need:
$n \cdot (m + 1)$ rounds

Observation: The effects of edges are not distributive!

Counterexample: $f = [\![x = x + y;]\!]^\sharp$

$$\begin{array}{lrcl} \text{Let} & D_1 & = & \{x \mapsto 2, y \mapsto 3\} \\ & D_2 & = & \{x \mapsto 3, y \mapsto 2\} \\ \text{Then} & f\,D_1 \sqcup f\,D_2 & = & \{x \mapsto 5, y \mapsto 3\} \sqcup \{x \mapsto 5, y \mapsto 2\} \\ & & = & \{x \mapsto 5, y \mapsto \top\} \\ & & \neq & \{x \mapsto \top, y \mapsto \top\} \\ & & = & f\,\{x \mapsto \top, y \mapsto \top\} \\ & & = & f\,(D_1 \sqcup D_2) \end{array}$$

We conclude:

The least solution $\mathcal{D}$ of the constraint system in general yields only an upper approximation of the MOP, i.e.,

$$\mathcal{D}^*[v] \quad \sqsubseteq \quad \mathcal{D}[v]$$

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$$\mathcal{D}^*[v] \sqsubseteq \mathcal{D}[v]$$

As an upper approximation, $\mathcal{D}[v]$ nonetheless describes the result of every program execution π that reaches v :

$$([\![\pi]\!] (\rho, \mu)) \;\; \Delta \;\; (\mathcal{D}[v])$$

whenever $[\![\pi]\!] (\rho, \mu)$ is defined

6 Interval Analysis

Constant propagation attempts to determine values of variables.

However, variables may take on several values during program execution.

So, *the value* of a variable will often be unknown.

Next attempt: determine an interval enclosing all possible values that a variable may take on during program execution at a program point.

Example:

$$
\begin{array}{l}
\textsf{for } (i = 0; i < 42; i{+}{+}) \\
\quad \textsf{if } (0 \leq i \wedge i < 42)\{ \\
\quad\quad A_1 = A + i; \\
\quad\quad M[A_1] = i; \\
\quad \}
\end{array}
$$

// A start address of an array

// if-statement does array-bounds check

Obviously, the inner check is superfluous.

Idea 1:

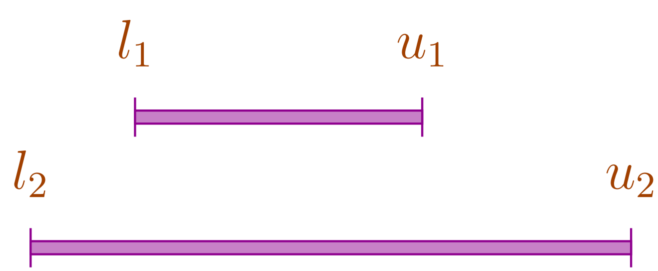
Determine for every variable x the tightest possible interval of potential values.

Abstract domain:

$$\mathbb{I} = \{[l, u] \mid l \in \mathbb{Z} \cup \{-\infty\}, u \in \mathbb{Z} \cup \{+\infty\}, l \leq u\}$$

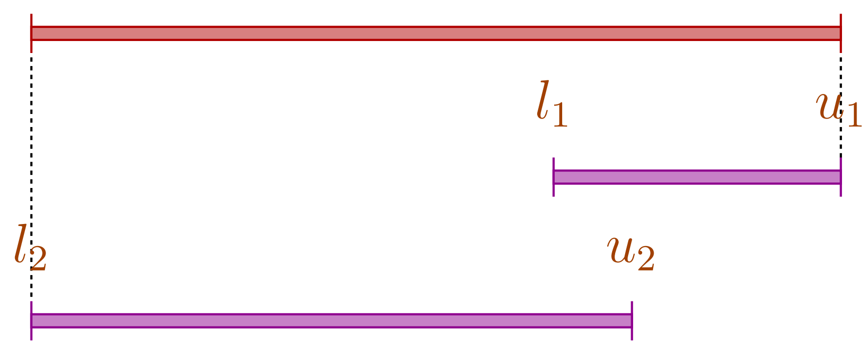
Partial order:

$$[l_1, u_1] \sqsubseteq [l_2, u_2] \qquad \text{iff} \qquad l_2 \leq l_1 \wedge u_1 \leq u_2$$



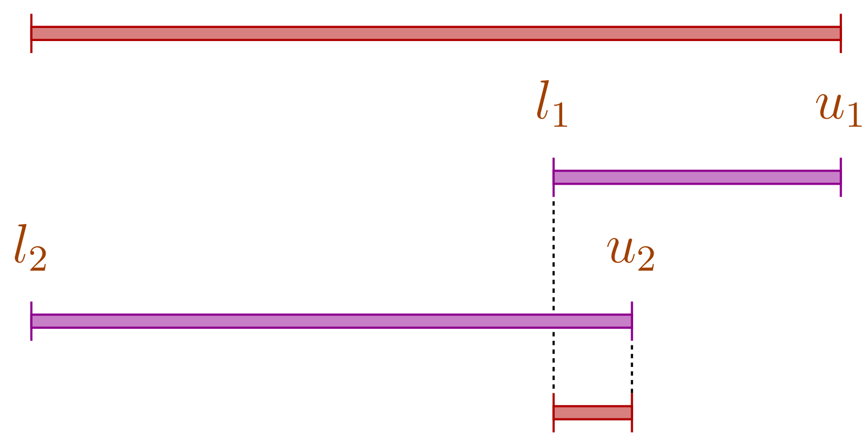
Thus:

$$[l_1, u_1] \sqcup [l_2, u_2] \quad = \quad [l_1 \sqcap l_2, u_1 \sqcup u_2]$$



Thus:

$$
\begin{array}{rcll}
[l_1, u_1] \sqcup [l_2, u_2] & = & [l_1 \sqcap l_2, u_1 \sqcup u_2] & \\
{[l_1, u_1] \sqcap [l_2, u_2]} & = & [l_1 \sqcup l_2, u_1 \sqcap u_2] & \quad \text{whenever } (l_1 \sqcup l_2) \leq (u_1 \sqcap u_2)
\end{array}
$$



Caveat:

- $\rightarrow$ $\mathbb{I}$ is not a complete lattice,
- $\rightarrow$ $\mathbb{I}$ has infinite ascending chains, e.g.,

$$[0,0] \sqsubset [0,1] \sqsubset [-1,1] \sqsubset [-1,2] \sqsubset \ldots$$

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Description Relation:

$$z \; \Delta \; [l,u] \qquad \text{iff} \qquad l \leq z \leq u$$

Concretization:

$$\gamma\,[l,u] = \{z \in \mathbb{Z} \mid l \leq z \leq u\}$$

Example:

$$\begin{aligned} \gamma\,[0,7] &= \{0,\ldots,7\} \\ \gamma\,[0,\infty] &= \{0,1,2,\ldots,\} \end{aligned}$$

Computing with intervals: Interval Arithmetic.

Addition:

$$\begin{aligned} [l_1,u_1] +^\sharp [l_2,u_2] &= [l_1+l_2, u_1+u_2] \qquad \text{where} \\ -\infty + \_ &= -\infty \\ +\infty + \_ &= +\infty \\ // & \quad -\infty + \infty \quad \text{cannot occur} \end{aligned}$$

Negation:

$$-^\sharp [l, u] \quad = \quad [-u, -l]$$

Multiplication:

$$\begin{aligned} [l_1, u_1] \; *^\sharp \; [l_2, u_2] &= [a, b] \qquad \text{where} \\ a &= l_1 l_2 \sqcap l_1 u_2 \sqcap u_1 l_2 \sqcap u_1 u_2 \\ b &= l_1 l_2 \sqcup l_1 u_2 \sqcup u_1 l_2 \sqcup u_1 u_2 \end{aligned}$$

Example:

$$\begin{aligned} [0, 2] \; *^\sharp \; [3, 4] &= [0, 8] \\ [-1, 2] \; *^\sharp \; [3, 4] &= [-4, 8] \\ [-1, 2] \; *^\sharp \; [-3, 4] &= [-6, 8] \\ [-1, 2] \; *^\sharp \; [-4, -3] &= [-8, 4] \end{aligned}$$

Division: $[l_1, u_1] \,/^\sharp\, [l_2, u_2] \;=\; [a, b]$

- If 0 is not contained in the interval of the denominator, then:

$$\begin{aligned} a &= l_1/l_2 \sqcap l_1/u_2 \sqcap u_1/l_2 \sqcap u_1/u_2 \\ b &= l_1/l_2 \sqcup l_1/u_2 \sqcup u_1/l_2 \sqcup u_1/u_2 \end{aligned}$$

- If: $l_2 \leq 0 \leq u_2$, we define:

$$[a, b] \;=\; [-\infty, +\infty]$$

Equality:

$$[l_1, u_1] ==^\sharp [l_2, u_2] \;=\; \begin{cases} \mathit{true} & \text{if} \quad l_1 = u_1 = l_2 = u_2 \\ \mathit{false} & \text{if} \quad u_1 < l_2 \vee u_2 < l_1 \\ \top & \text{otherwise} \end{cases}$$

Equality:

$$[l_1, u_1] ==^\sharp [l_2, u_2] \;=\; \begin{cases} \mathit{true} & \text{if} \quad l_1 = u_1 = l_2 = u_2 \\ \mathit{false} & \text{if} \quad u_1 < l_2 \vee u_2 < l_1 \\ \top & \text{otherwise} \end{cases}$$

Example:

$$\begin{array}{rcl} [42, 42] ==^\sharp [42, 42] & = & \mathit{true} \\ [0, 7] ==^\sharp [0, 7] & = & \top \\ [1, 2] ==^\sharp [3, 4] & = & \mathit{false} \end{array}$$

Less:

$$[l_1, u_1] <^\sharp [l_2, u_2] = \begin{cases} true & \text{if} \quad u_1 < l_2 \\ false & \text{if} \quad u_2 \leq l_1 \\ \top & \text{otherwise} \end{cases}$$

Less:

$$[l_1, u_1] \ <^\sharp \ [l_2, u_2] \ = \ \begin{cases} true & \text{if} \quad u_1 < l_2 \\ false & \text{if} \quad u_2 \leq l_1 \\ \top & \text{otherwise} \end{cases}$$

Example:

$$\begin{aligned} [1, 2] <^\sharp [9, 42] &= true \\ [0, 7] \ <^\sharp \ [0, 7] &= \top \\ [3, 4] \ <^\sharp \ [1, 2] &= false \end{aligned}$$

By means of $\mathbb{I}$ we construct the complete lattice:

$$\mathbb{D}_{\mathbb{I}} = (\,\mathit{Vars} \to \mathbb{I})_{\perp}$$

Description Relation:

$$\rho \;\;\Delta\;\; D \qquad \text{iff} \qquad D \neq \perp \quad \wedge \quad \forall\, x \in \mathit{Vars} : \;(\rho\, x) \;\;\Delta\;\; (D\;x)$$

The abstract evaluation of expressions is defined analogously to constant propagation. We have:

$$([\![e]\!]\;\rho) \;\;\Delta\;\; ([\![e]\!]^{\sharp}\;D) \qquad \text{whenever} \qquad \rho \;\;\Delta\;\; D$$

The Effects of Edges:

$$
\begin{array}{lcl}
[\![;]\!]^\sharp\, D & = & D \\
[\![x = e;]\!]^\sharp\, D & = & D \oplus \{x \mapsto [\![e]\!]^\sharp\, D\} \\
[\![x = M[e];]\!]^\sharp\, D & = & D \oplus \{x \mapsto \top\} \\
[\![M[e_1] = e_2;]\!]^\sharp\, D & = & D \\
[\![\text{true}\,(e)]\!]^\sharp\, D & = & \begin{cases} \bot & \text{if} \qquad \text{definitely false} \\ D & \text{otherwise} \qquad \text{possibly true} \end{cases} \\
[\![\text{false}\,(e)]\!]^\sharp\, D & = & \begin{cases} D & \text{if} \qquad \text{possibly false} \\ \bot & \text{otherwise} \qquad \text{definitely true} \end{cases}
\end{array}
$$

... given that $\quad D \neq \bot$

Better Exploitation of Conditions:

$$[\![\text{Pos}\,(e)]\!]^\sharp\, D \quad = \quad \begin{cases} \bot & \text{if} \quad false = [\![e]\!]^\sharp\, D \\ D_1 & \text{otherwise} \end{cases}$$

where :

$$D_1 \quad = \quad \begin{cases} D \oplus \{x \mapsto (D\,x) \sqcap ([\![e_1]\!]^\sharp\, D)\} & \text{if } e \equiv x == e_1 \\ D \oplus \{x \mapsto (D\,x) \sqcap [-\infty, u]\} & \text{if } e \equiv x \leq e_1, [\![e_1]\!]^\sharp\, D = [\_, u] \\ D \oplus \{x \mapsto (D\,x) \sqcap [l, \infty]\} & \text{if } e \equiv x \geq e_1, [\![e_1]\!]^\sharp\, D = [l, \_] \end{cases}$$

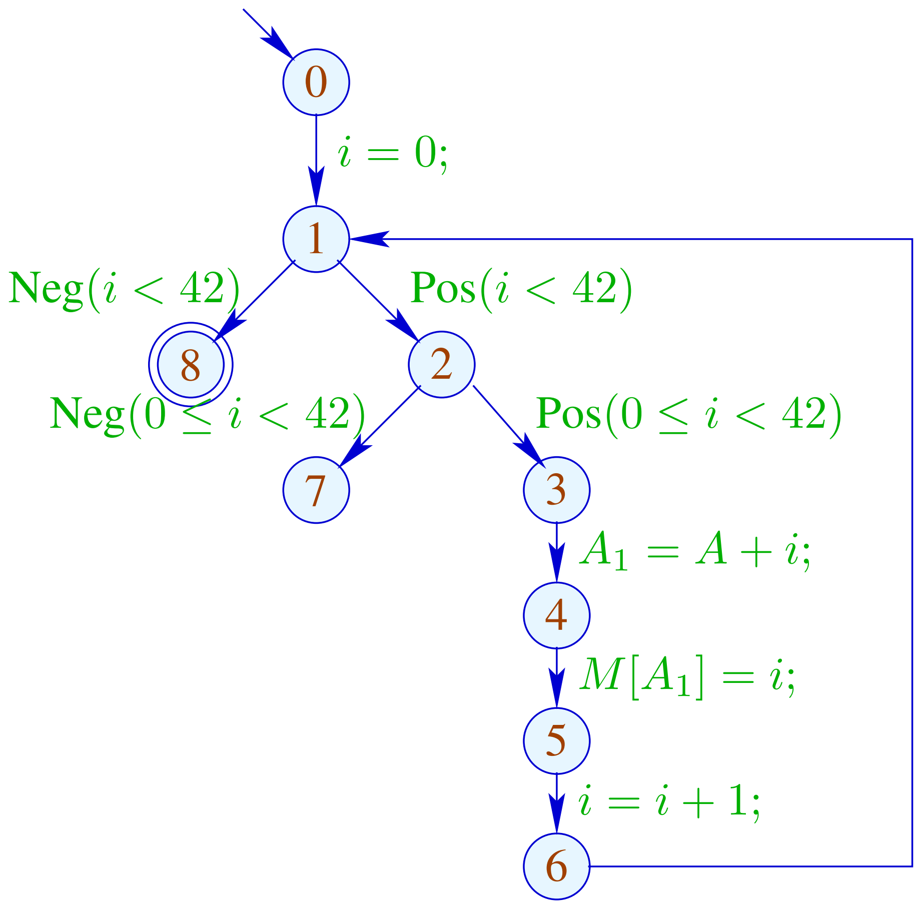
Better Exploitation of Conditions (cont.):

$$[\![\text{Neg}\,(e)]\!]^\sharp\,D \quad = \quad \begin{cases} \bot & \text{if} \quad \mathit{false} \not\sqsubseteq [\![e]\!]^\sharp\,D \\ D_1 & \text{otherwise} \end{cases}$$

where :

$$D_1 \quad = \quad \begin{cases} D \oplus \{x \mapsto (D\,x) \sqcap ([\![e_1]\!]^\sharp\,D)\} & \text{if } e \equiv x \neq e_1 \\ D \oplus \{x \mapsto (D\,x) \sqcap [-\infty, u]\} & \text{if } e \equiv x > e_1, [\![e_1]\!]^\sharp\,D = [\_, u] \\ D \oplus \{x \mapsto (D\,x) \sqcap [l, \infty]\} & \text{if } e \equiv x < e_1, [\![e_1]\!]^\sharp\,D = [l, \_] \end{cases}$$

Example:



	i	
	l	u
0	$-\infty$	$+\infty$
1	0	42
2	0	41
3	0	41
4	0	41
5	0	41
6	1	42
7	$\perp$	
8	42	42

Problem:

→ The solution can be computed with RR-iteration — after about 42 rounds.

→ On some programs, iteration may never terminate.

Idea: Widening

Accelerate the iteration — at the cost of precision

Formalization of the Approach:

Let $x_i \sqsupseteq f_i\,(x_1,\ldots,x_n)\;,\quad i = 1,\ldots,n$ (1)

denote a system of constraints over $\mathbb{D}$

Define an accumulating iteration:

$$x_i \;=\; x_i \sqcup f_i\,(x_1,\ldots,x_n)\;,\quad i = 1,\ldots,n \qquad (2)$$

We obviously have:

(a) $\underline{x}$ is a solution of (1) iff $\underline{x}$ is a solution of (2).

(b) The function $G : \mathbb{D}^n \to \mathbb{D}^n$ with
$G\,(x_1,\ldots,x_n) = (y_1,\ldots,y_n)\;,\quad y_i = x_i \sqcup f_i\,(x_1,\ldots,x_n)$
is increasing, i.e., $\underline{x} \sqsubseteq G\,\underline{x}$ for all $\underline{x} \in \mathbb{D}^n$.

(c) The sequence $G^k \underline{\perp}$, $k \geq 0$, is an ascending chain:

$$\underline{\perp} \sqsubseteq G \underline{\perp} \sqsubseteq \ldots \sqsubseteq G^k \underline{\perp} \sqsubseteq \ldots$$

(d) If $G^k \underline{\perp} = G^{k+1} \underline{\perp} = \underline{y}$, then $\underline{y}$ is a solution of (1).

(e) If $\mathbb{D}$ has infinite strictly ascending chains, then (d) is not yet sufficient ...

but: we could consider the modified system of equations:

$$x_i = x_i \sqcup f_i(x_1, \ldots, x_n) \;, \quad i = 1, \ldots, n \qquad (3)$$

for a binary operation widening:

$$\sqcup : \mathbb{D}^2 \to \mathbb{D} \qquad \text{with} \qquad v_1 \sqcup v_2 \sqsubseteq v_1 \sqcup v_2$$

(RR)-iteration for (3) still will compute a solution of (1)

... for Interval Analysis:

- The complete lattice is: $\mathbb{D}_{\mathbb{I}} \;=\; (\mathit{Vars} \to \mathbb{I})_\perp$
- the widening $\sqcup$ is defined by:

$$\perp \sqcup D \;=\; D \sqcup \perp \;=\; D \qquad \text{and for} \quad D_1 \neq \perp \neq D_2:$$

$$
\begin{array}{rcl}
(D_1 \sqcup D_2)\, x & = & (D_1\, x) \sqcup (D_2\, x) \qquad \text{where} \\
{[l_1, u_1]} \sqcup [l_2, u_2] & = & [l, u] \qquad \text{with} \\
l & = & \begin{cases} l_1 & \text{if} \quad l_1 \leq l_2 \\ -\infty & \text{otherwise} \end{cases} \\
u & = & \begin{cases} u_1 & \text{if} \quad u_1 \geq u_2 \\ +\infty & \text{otherwise} \end{cases}
\end{array}
$$

$\Longrightarrow$ $\sqcup$ is not commutative !!!

Example:

$$
\begin{aligned}
[0,2] \sqcup [1,2] &= [0,2] \\
[1,2] \sqcup [0,2] &= [-\infty,2] \\
[1,5] \sqcup [3,7] &= [1,+\infty]
\end{aligned}
$$

→ Widening returns larger values more quickly.

→ It should be constructed in such a way that termination of iteration is guaranteed.

→ For interval analysis, widening bounds the number of iterations by:

$$\#\mathit{points} \cdot (1 + 2 \cdot \#\mathit{Vars})$$

Conclusion:

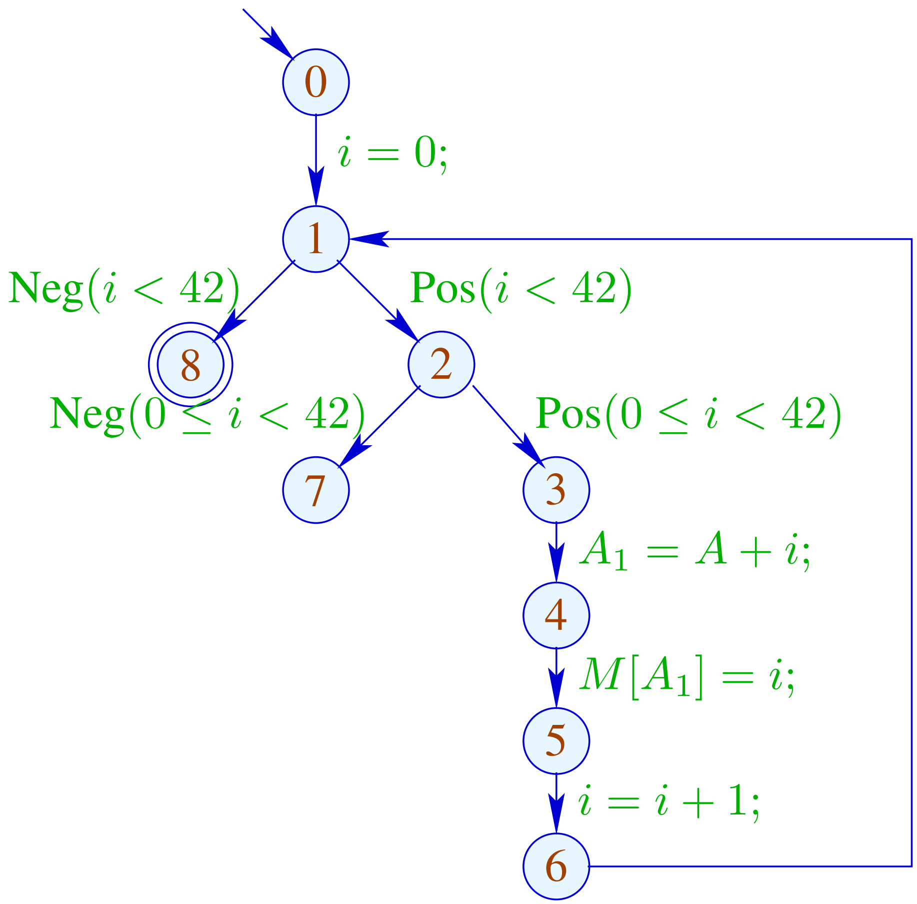
- In order to determine a solution of (1) over a complete lattice with infinite ascending chains, we define a suitable widening and then solve (3)
- Caveat: The construction of suitable widenings is a dark art !!!

  Often $\sqcup$ is chosen dynamically during iteration such that

  $\rightarrow$ the abstract values do not get too complicated;

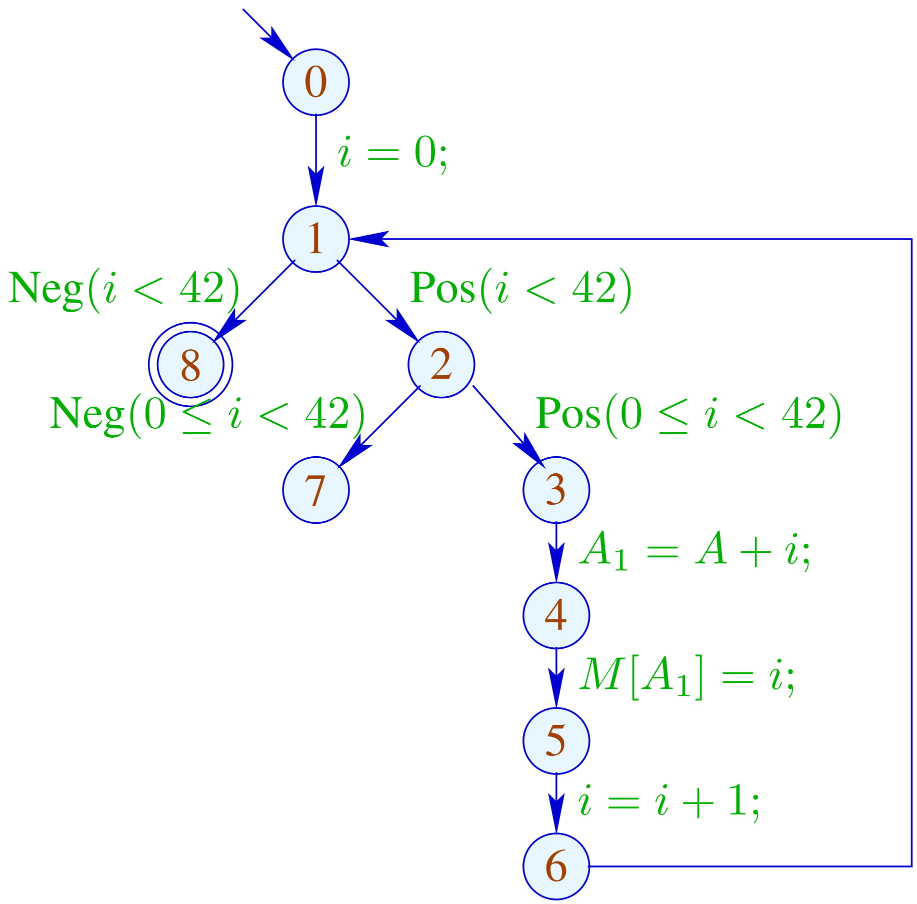
  $\rightarrow$ the number of updates remains bounded ...

Our Example:



	1	
	l	u
0	$-\infty$	$+\infty$
1	0	0
2	0	0
3	0	0
4	0	0
5	0	0
6	1	1
7	$\perp$	
8	$\perp$	

Our Example:



	1		2		3	
	l	u	l	u	l	u
0	$-\infty$	$+\infty$	$-\infty$	$+\infty$	dito	
1	0	0	0	$+\infty$		
2	0	0	0	$+\infty$		
3	0	0	0	$+\infty$		
4	0	0	0	$+\infty$		
5	0	0	0	$+\infty$		
6	1	1	1	$+\infty$		
7	$\bot$		42	$+\infty$		
8	$\bot$		42	$+\infty$		

7 Removing superfluous computations

A computation may be superfluous because

- the result is already available, $\longrightarrow$ available-expression analysis, or
- the result is not needed $\longrightarrow$ live-variable analysis.

7.1 Redundant computations

Idea:

If an expression at a program point is guaranteed to be computed to the value it had before, then

$\rightarrow$ store this value after the first computation;

$\rightarrow$ replace every further computation through a look-up

Question to be answered by static analysis: Is an expression available?

Problem: Identify sources of redundant computations!

Example:

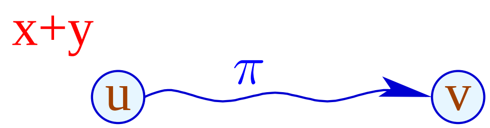
$$\begin{array}{lrcl} & z & = & 1; \\ & y & = & M[17]; \\ A: & x_1 & = & \boxed{y+z}; \\ & & \dots & \\ B: & x_2 & = & \boxed{y+z}; \end{array}$$

B is a redundant computation of the value of $\boxed{y+z}$, if

(1) A is always executed before B; and

(2) y and z at B have the same values as at A

Situation: The value of $x + y$ is computed at program point u



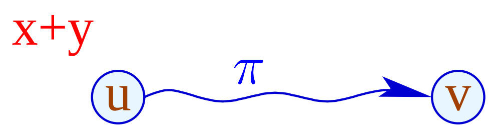
and a computation along path π reaches v where it evaluates again $x + y$

....

If x and y have not been modified in π, then evaluation of $x + y$ at v returns the same value as evaluation at u.

This property can be checked at every edge in π.

Situation: The value of $x + y$ is computed at program point u



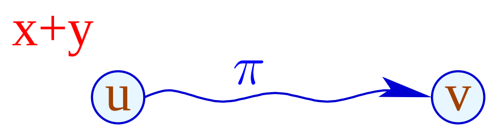
and a computation along path π reaches v where it evaluates again $x + y$ .... If x and y have not been modified in π, then evaluation of $x + y$ at v is known to return the same value as evaluation at u

This property can be checked at every edge in π.

More efficient: Do this check for all expressions occurring in the program in parallel.

Assume that the expressions $A = \{e_1, \ldots, e_r\}$ are available at u.

Situation: The value of $x + y$ is computed at program point u



and a computation along path π reaches v where it evaluates again $x + y$ .... If x and y have not been modified in π, then evaluation of $x + y$ at v must return the same value as evaluation at u.

This property can be checked at every edge in π.

More efficient: Do this check for all expressions occurring in the program in parallel.

Assume that the expressions $A = \{e_1, \ldots, e_r\}$ are available at u.

Every edge k transforms this set into a set $[\![k]\!]^\sharp\, A$ of expressions whose values are available after execution of k.

$[\![k]\!]^\sharp\, A$ is the (abstract) edge effect associated with k

These edge effects can be composed to the effect of a path $\pi = k_1 \ldots k_r$:

$$[\![\pi]\!]^\sharp = [\![k_r]\!]^\sharp \circ \ldots \circ [\![k_1]\!]^\sharp$$

These edge effects can be composed to the effect of a path $\pi = k_1 \ldots k_r$:

$$[\![\pi]\!]^\sharp = [\![k_r]\!]^\sharp \circ \ldots \circ [\![k_1]\!]^\sharp$$

The effect $[\![k]\!]^\sharp$ of an edge $k = (u, lab, v)$ only depends on the label lab, i.e., $[\![k]\!]^\sharp = [\![lab]\!]^\sharp$

These edge effects can be composed to the effect of a path $\pi = k_1 \ldots k_r$:

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The effect $[\![k]\!]^\sharp$ of an edge $k = (u, lab, v)$ only depends on the label lab, i.e., $[\![k]\!]^\sharp = [\![lab]\!]^\sharp$ where:

$$\begin{array}{lll} [\![;]\!]^\sharp\, A & = & A \\ [\![true(e)]\!]^\sharp\, A & = & [\![false(e)]\!]^\sharp\, A \quad = \quad A \cup \{e\} \\ [\![x = e;]\!]^\sharp\, A & = & (A \cup \{e\}) \backslash Expr_x \qquad \text{where} \end{array}$$

$Expr_x$ all expressions that contain x

$$\begin{array}{lll} [\![x = M[e];]\!]^\sharp\, A & = & (A \cup \{e\}) \backslash Expr_x \\ [\![M[e_1] = e_2;]\!]^\sharp\, A & = & A \cup \{e_1, e_2\} \end{array}$$

- $\rightarrow$ An expression is available at v if it is available along all paths π to v.
- $\rightarrow$ For every such path π, the analysis determines the set of expressions that are available along π.
- $\rightarrow$ Initially at program start, nothing is available.
- $\rightarrow$ The analysis computes the intersection of the availability sets as safe information.

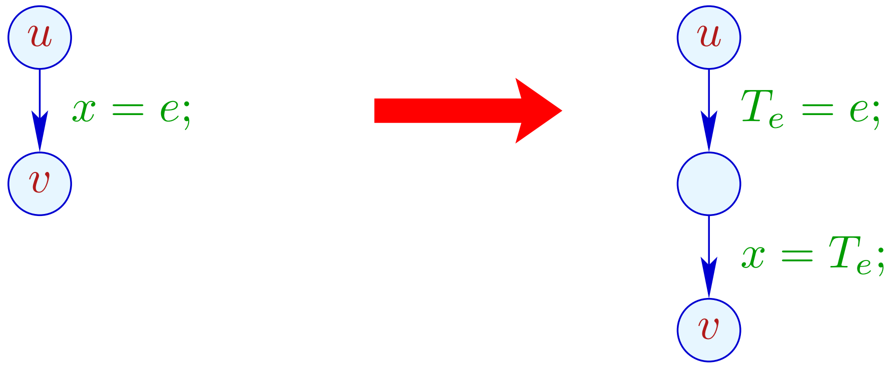
$\Longrightarrow$ For each node v, we need the set:

$$\mathcal{A}[v] \quad = \quad \bigcap \{ [\![\pi]\!]^\sharp \emptyset \mid \pi : \mathit{start} \rightarrow^* v \}$$

How does a compiler exploit this information?

Transformation UT (unique temporaries):

We provide a novel register T_e as storage for the values of e:



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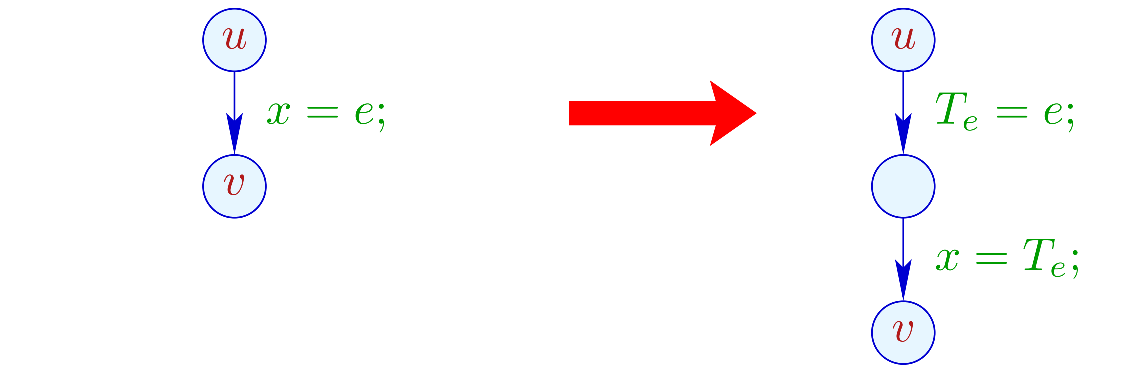
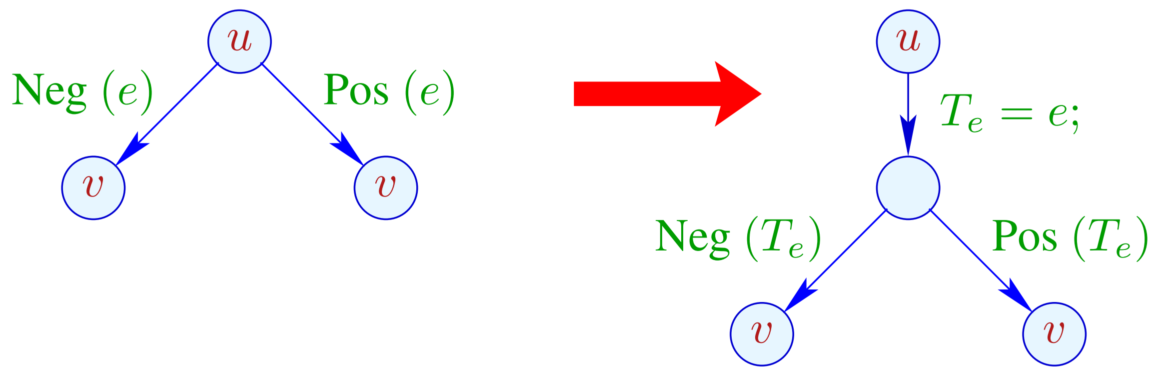
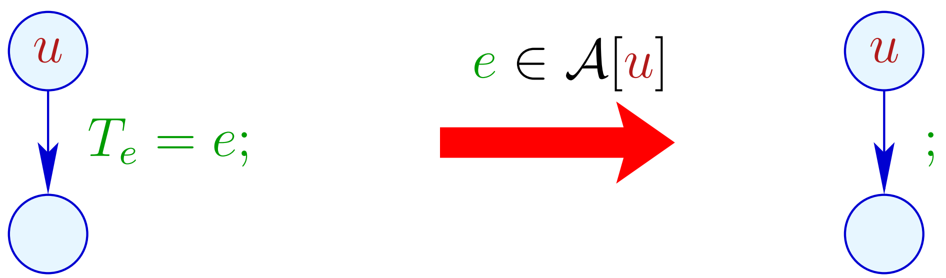




... analogously for $R = M[e];$ and $M[e_1] = e_2;$.

Transformation AEE (available expression elimination):

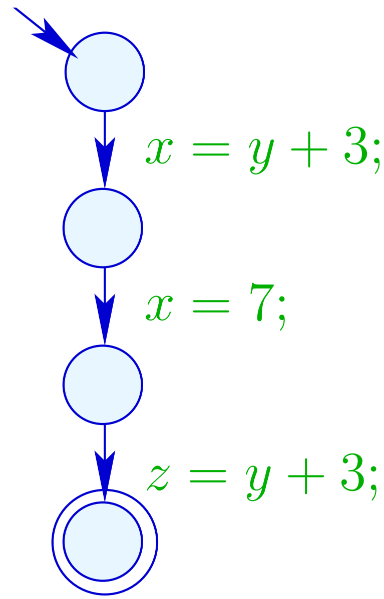
If e is available at program point u, then e need not be re-evaluated:



We replace the assignment with Nop.

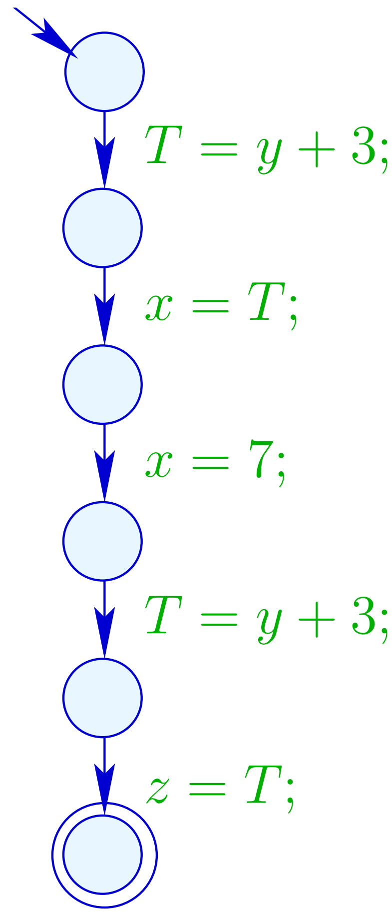
Example:

$$\begin{aligned} x &= y + 3; \\ x &= 7; \\ z &= y + 3; \end{aligned}$$



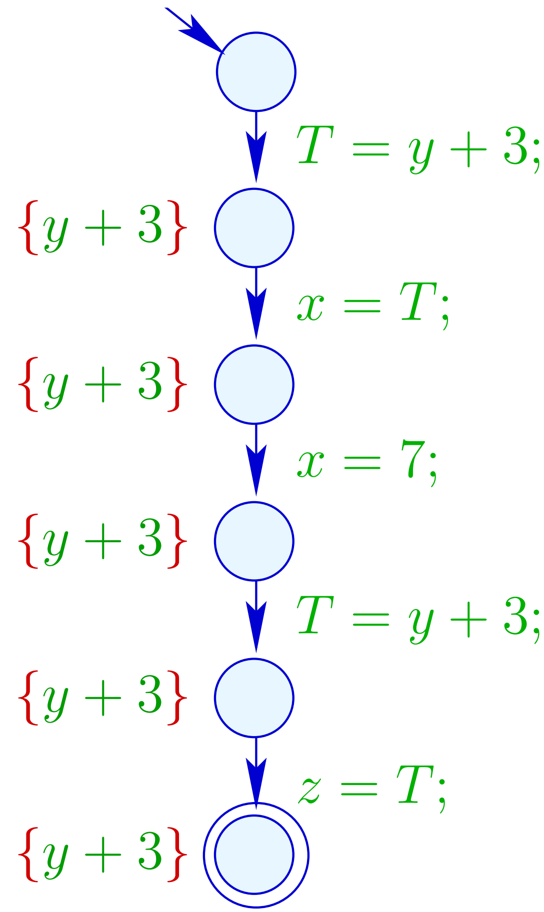
Example:

$$x = y + 3;$$
$$x = 7;$$
$$z = y + 3;$$



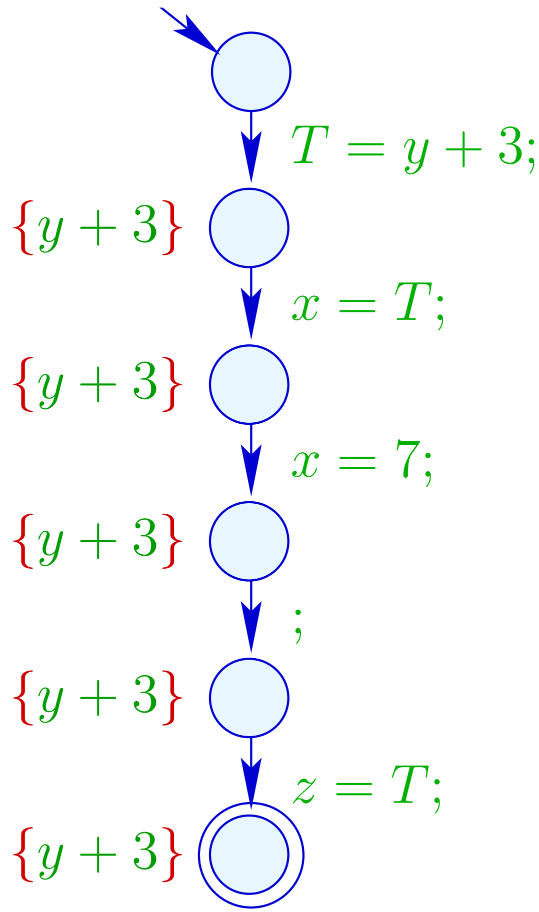
Example:

$$
\begin{aligned}
x &= y+3; \\
x &= 7; \\
z &= y+3;
\end{aligned}
$$



Example:

$$
\begin{aligned}
x &= y + 3; \\
x &= 7; \\
z &= y + 3;
\end{aligned}
$$



Warning:

Transformation UT is only meaningful for assignments $x = e;$ where:

- $\rightarrow$ $x \notin Vars(e)$; why?
- $\rightarrow$ $e \notin Vars$; why?
- $\rightarrow$ the evaluation of e is non-trivial; why?

Warning:

Transformation UT is only meaningful for assignments $x = e;$ where:

- $\rightarrow$ $x \notin Vars(e)$; otherwise e is not available afterwards.
- $\rightarrow$ $e \notin Vars$; otherwise values are shuffled around
- $\rightarrow$ the evaluation of e is non-trivial; otherwise the efficiency of the code is decreased.

Open question ...

Question:

How do we compute $\mathcal{A}[u]$ for every program point u?

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How can we compute $\mathcal{A}[u]$ for every program point? u

We collect all constraints on the values of $\mathcal{A}[u]$ into a system of constraints:

$$\begin{array}{lll} \mathcal{A}[start] & \subseteq & \emptyset \\ \mathcal{A}[v] & \subseteq & [\![k]\!]^\sharp\,(\mathcal{A}[u]) \qquad\qquad k = (u, \_, v) \quad \text{edge} \end{array}$$

Why $\subseteq$?

Question:

How can we compute $\mathcal{A}[u]$ for every program point? u

Idea:

We collect all constraints on the values of $\mathcal{A}[u]$ into a system of constraints:

$$
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\mathcal{A}[start] & \subseteq & \emptyset \\
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\end{array}
$$

Why $\subseteq$?

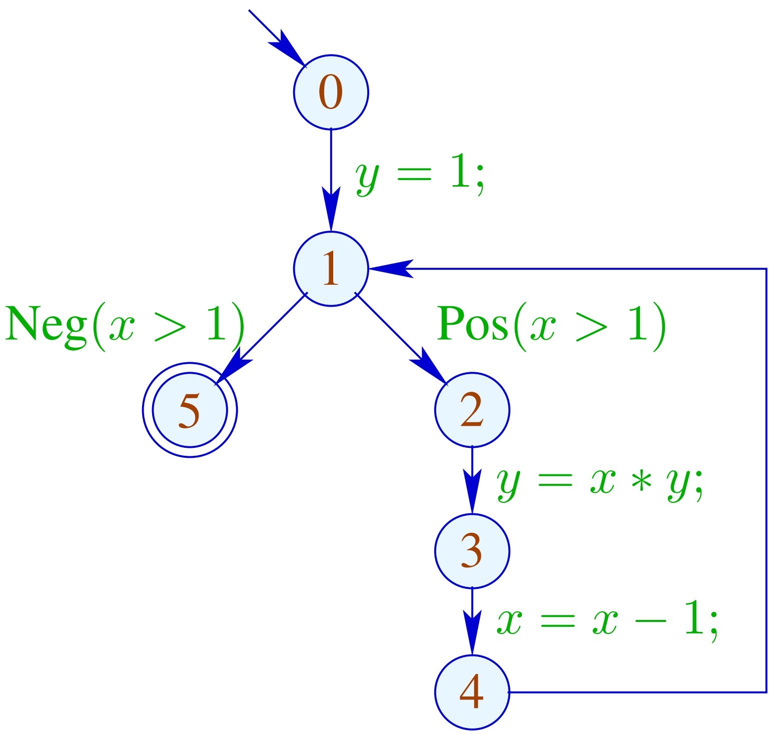
Then combine all constraints for each variable v by applying the least-upper-bound operator $\longrightarrow$

$$
\mathcal{A}[v] \quad \subseteq \quad \bigcap\{[\![k]\!]^\sharp\,(\mathcal{A}[u]) \mid k = (u, \_, v) \;\; \text{edge}\}
$$

Wanted:

- a greatest solution (why greatest?)
- an algorithm that computes this solution

Example:



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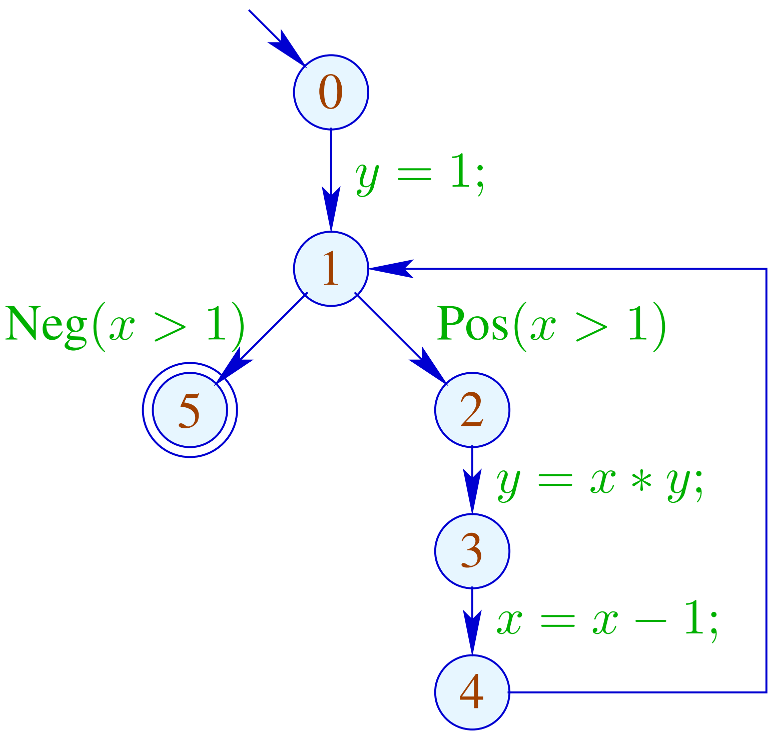

$$\mathcal{A}[0] \quad \subseteq \quad \emptyset$$

Wanted:

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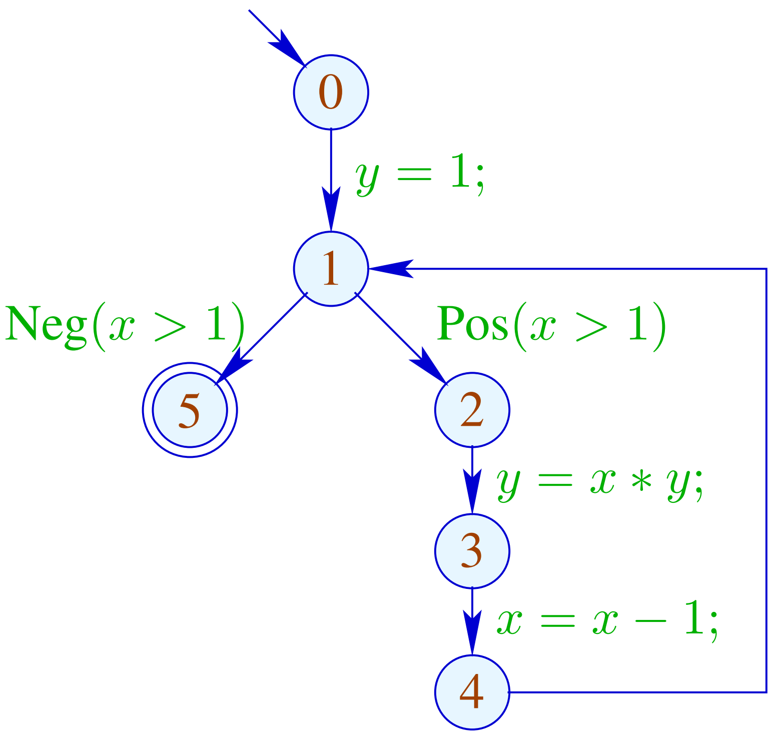
Example:



$$\begin{array}{lll} \mathcal{A}[0] & \subseteq & \emptyset \\ \mathcal{A}[1] & \subseteq & (\mathcal{A}[0] \cup \{1\}) \backslash Expr_y \\ \mathcal{A}[1] & \subseteq & \mathcal{A}[4] \end{array}$$

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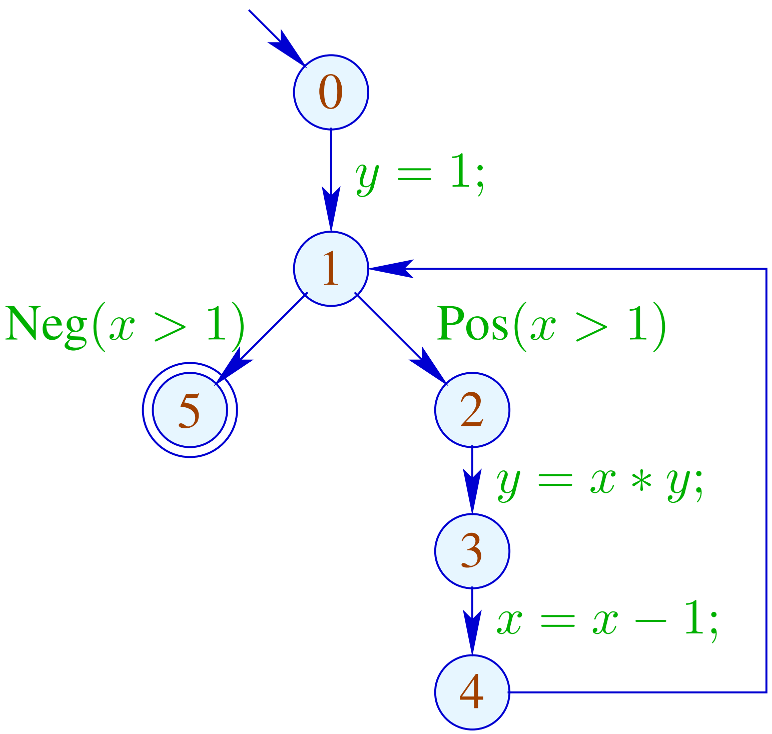

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\end{array}$$

Wanted:

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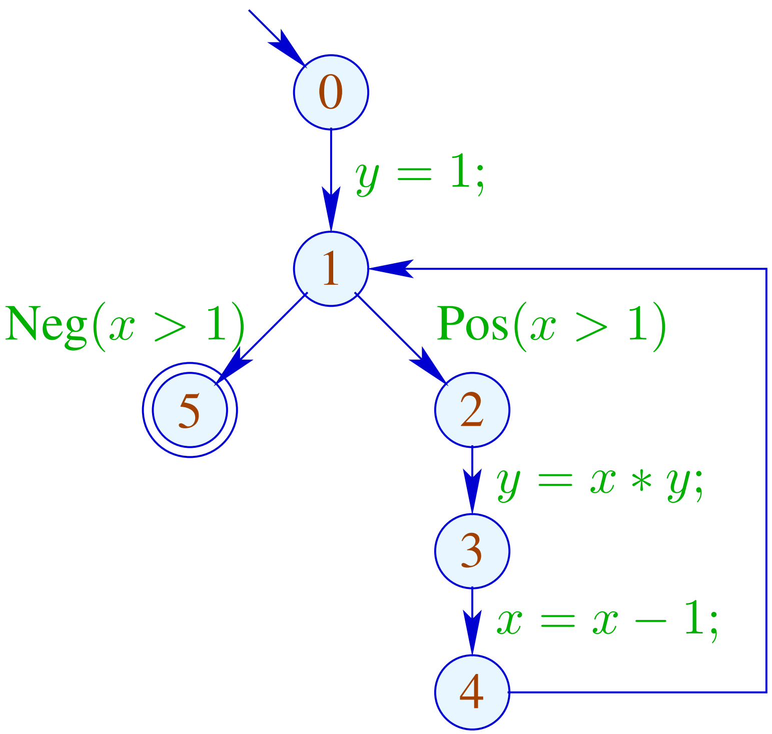
Example:



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$$

Wanted:

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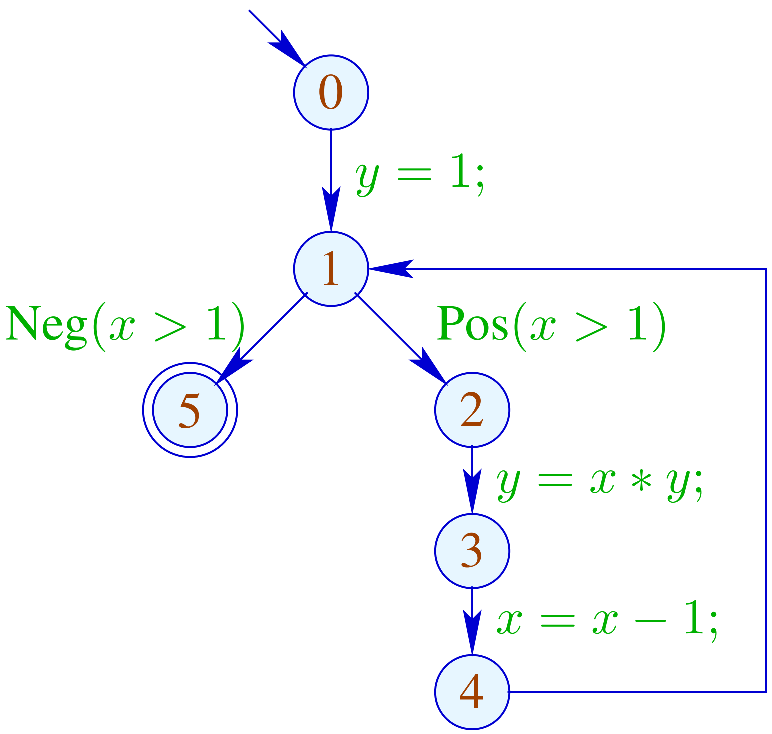
Example:



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\mathcal{A}[4] & \subseteq & (\mathcal{A}[3] \cup \{x - 1\}) \backslash Expr_x
\end{array}
$$

Wanted:

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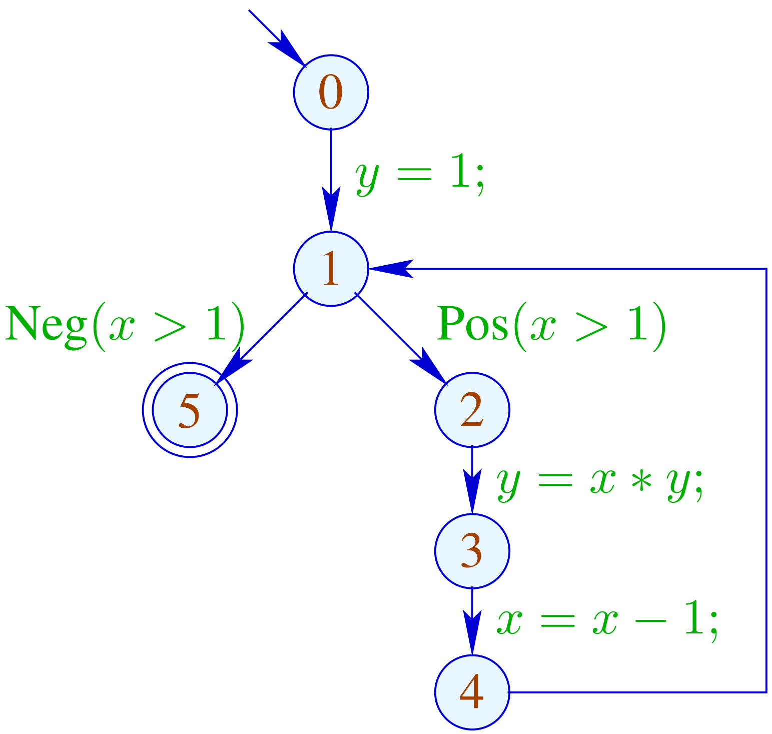
Example:



$$
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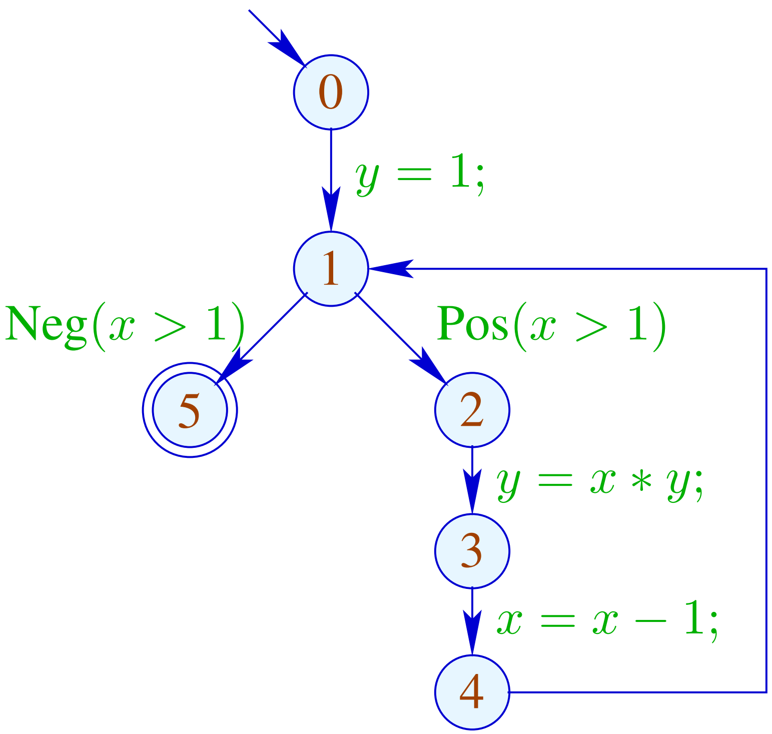
\mathcal{A}[4] & \subseteq & (\mathcal{A}[3] \cup \{x - 1\}) \backslash Expr_x \\
\mathcal{A}[5] & \subseteq & \mathcal{A}[1] \cup \{x > 1\}
\end{array}
$$

Wanted:

- a greatest solution,
- an algorithm that computes this solution.

Example:



Solution:

$$
\begin{array}{lll}
\mathcal{A}[0] & = & \emptyset \\
\mathcal{A}[1] & = & \{1\} \\
\mathcal{A}[2] & = & \{1, x > 1\} \\
\mathcal{A}[3] & = & \{1, x > 1\} \\
\mathcal{A}[4] & = & \{1\} \\
\mathcal{A}[5] & = & \{1, x > 1\}
\end{array}
$$

Observation:

- Again, the possible values for $\mathcal{A}[u]$ form a complete lattice:

$$\mathbb{D} = 2^{Expr} \quad \text{with} \quad B_1 \sqsubseteq B_2 \quad \text{iff} \quad B_1 \supseteq B_2$$

- The order on the lattice elements indicates what is better information,
  more available expressions may allow more optimizations

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- The order on the lattice elements indicates what is better information,
  more available expressions may allow more optimizations
- The functions $[\![k]\!]^\sharp : \mathbb{D} \to \mathbb{D}$ have the form $\quad f_i\, x = a_i \cap x \cup b_i$.
  They are called *gen/kill* functions—$\cap$ kills, $\cup$ generates.
- they are monotonic, i.e.,

$$[\![k]\!]^\sharp(B_1) \sqsubseteq [\![k]\!]^\sharp(B_2) \quad \text{iff} \quad B_1 \sqsubseteq B_2$$

The operations "$\circ$", "$\sqcup$" and "$\sqcap$" can be explicitly defined by:

$$
\begin{aligned}
(f_2 \circ f_1)\, x &= \boxed{a_1 \cap a_2} \cap x \cup \boxed{a_2 \cap b_1 \cup b_2} \\
(f_1 \sqcup f_2)\, x &= \boxed{(a_1 \cup a_2)} \cap x \cup \boxed{b_1 \cup b_2} \\
(f_1 \sqcap f_2)\, x &= \boxed{(a_1 \cup b_1) \cap (a_2 \cup b_2)} \cap x \cup \boxed{b_1 \cap b_2}
\end{aligned}
$$

7.2 Removing Assignments to Dead Variables

Example:

$$\begin{array}{ll} 1: & x = y + 2; \\ 2: & y = 5; \\ 3: & x = y + 3; \end{array}$$

The value of x at program points $1, 2$ is overwritten before it can be used.

Therefore, we call the variable x dead at these program points.

Note:

→ Assignments to dead variables can be removed.

→ Such inefficiencies may originate from other transformations.

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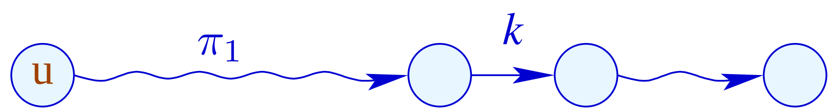
$\rightarrow$ Assignments to dead variables can be removed.

$\rightarrow$ Such inefficiencies may originate from other transformations.

Formal Definition:

The variable x is called live at u along a path π starting at u if π can be decomposed into $\pi = \pi_1\, k\, \pi_2$ such that:

- k is a use of x and
- π_1 does not contain a definition of x.

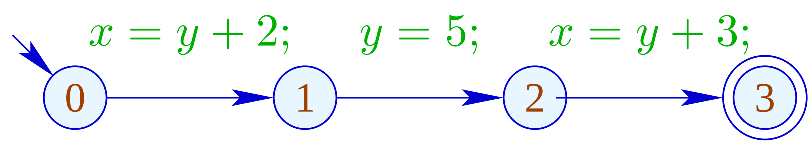


Thereby, the set of all defined or used variables at an edge $k = (\_, lab, \_)$ is defined by

lab	used	defined
;	$\emptyset$	$\emptyset$
$\text{true}(e)$	$Vars(e)$	$\emptyset$
$\text{false}(e)$	$Vars(e)$	$\emptyset$
$x = e;$	$Vars(e)$	$\{x\}$
$x = M[e];$	$Vars(e)$	$\{x\}$
$M[e_1] = e_2;$	$Vars(e_1) \cup Vars(e_2)$	$\emptyset$

A variable x which is not live at u along π is called dead at u along π.

Example:



Then we observe:

	live	dead
0	$\{y\}$	$\{x\}$
1	$\emptyset$	$\{x, y\}$
2	$\{y\}$	$\{x\}$
3	$\emptyset$	$\{x, y\}$

The variable x is live at u if x is live at u along some path to the exit . Otherwise, x is called dead at u .

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Question:

How can the sets of all dead/live variables be computed for every u?

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Question:

How can the sets of all dead/live variables be computed for every u?

Idea:

For every edge $k = (u, \_, v)$, define a function $[\![k]\!]^\sharp$ which transforms the set of variables that are live at v into the set of variables that are live at u.

Note: Edge transformers go "backwards"!

Let $\mathbb{L} = 2^{Vars}$.

For $k = (\_, lab, \_)$, define $[\![k]\!]^\sharp = [\![lab]\!]^\sharp$ by:

$$
\begin{array}{lcl}
[\![;]\!]^\sharp\, L & = & L \\
[\![\text{true}(e)]\!]^\sharp\, L & = & [\![\text{false}(e)]\!]^\sharp\, L \quad = \quad L \cup Vars(e) \\
[\![x = e;]\!]^\sharp\, L & = & (L \backslash \{x\}) \cup Vars(e) \\
[\![x = M[e];]\!]^\sharp\, L & = & (L \backslash \{x\}) \cup Vars(e) \\
[\![M[e_1] = e_2;]\!]^\sharp\, L & = & L \cup Vars(e_1) \cup Vars(e_2)
\end{array}
$$

Let $\mathbb{L} = 2^{Vars}$.

For $k = (\_, lab, \_)$, define $[\![k]\!]^\sharp = [\![lab]\!]^\sharp$ by:

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[\![\text{true}(e)]\!]^\sharp\, L & = & [\![\text{false}(e)]\!]^\sharp\, L \quad = \quad L \cup Vars(e) \\
[\![x = e;]\!]^\sharp\, L & = & (L \backslash \{x\}) \cup Vars(e) \\
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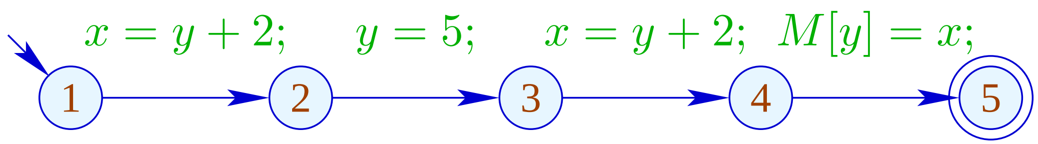
[\![M[e_1] = e_2;]\!]^\sharp\, L & = & L \cup Vars(e_1) \cup Vars(e_2)
\end{array}
$$

$[\![k]\!]^\sharp$ can again be composed to the effects of $[\![\pi]\!]^\sharp$ of paths $\pi = k_1 \ldots k_r$ by:

$$[\![\pi]\!]^\sharp = [\![k_1]\!]^\sharp \circ \ldots \circ [\![k_r]\!]^\sharp$$

We verify that these definitions are meaningful



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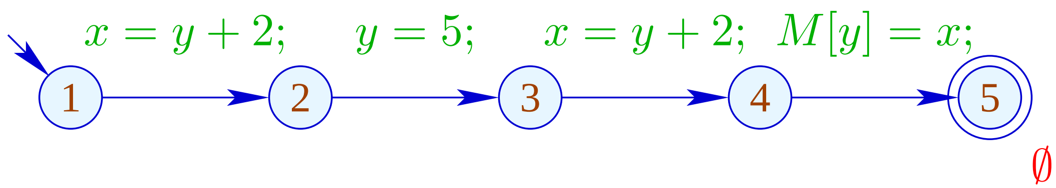

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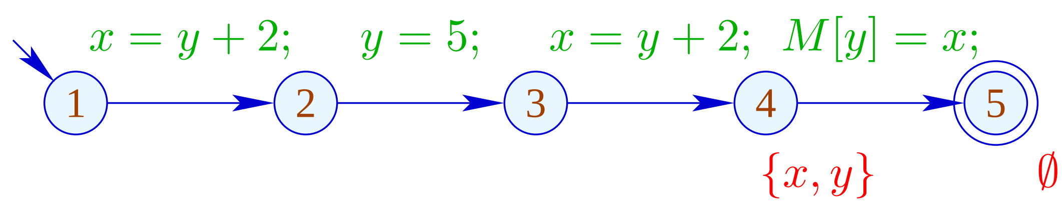

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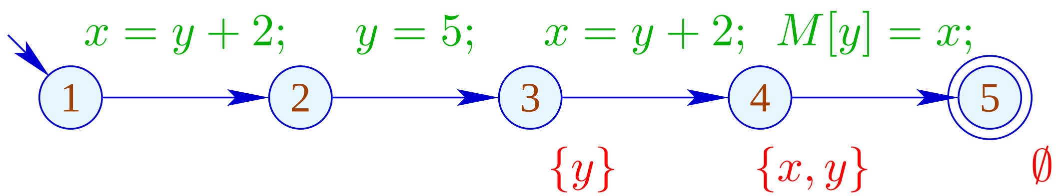

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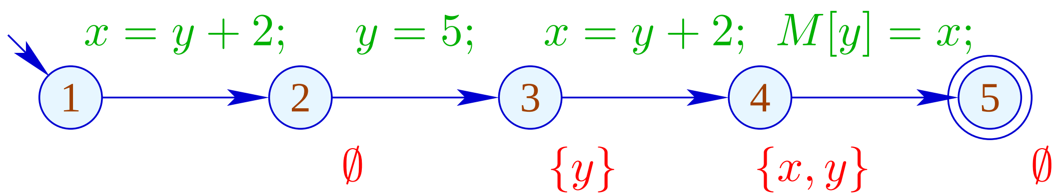

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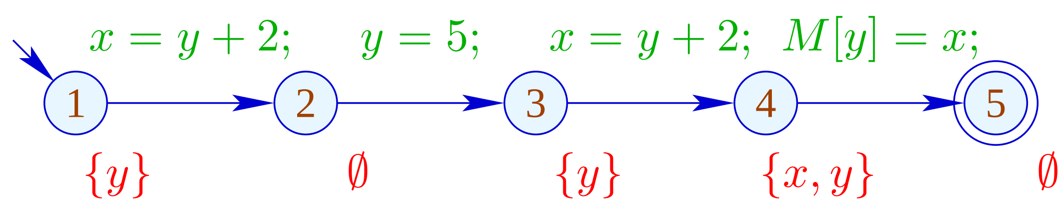


A variable is live at a program point u if there is at least one path from u to program exit on which it is live.

The set of variables which are live at u therefore is given by:

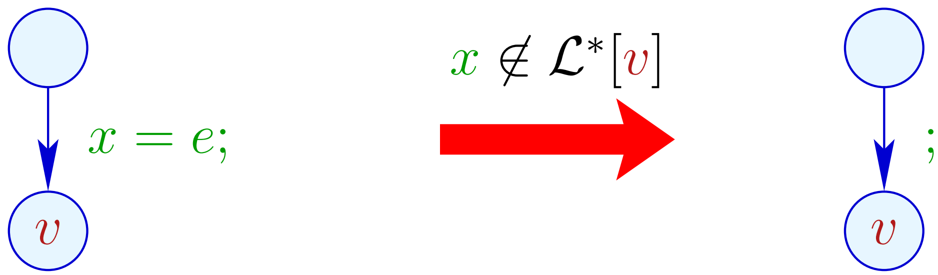
$$\mathcal{L}^*[u] \;=\; \bigcup\{[\![\pi]\!]^\sharp\emptyset \mid \pi : u \rightarrow^* \mathit{stop}\}$$

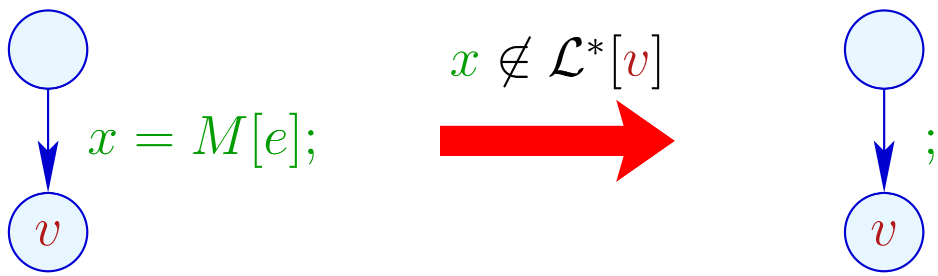
No variables are assumed to be live at program exit.

As partial order for $\mathbb{L}$ we use $\sqsubseteq \, = \, \subseteq$. why?

So, the least upper bound is $\bigcup$. why?

Transformation DE (Dead assignment Elimination):





Correctness Proof:

- → **Correctness of the effects of edges:** If L is the set of variables which are live at the exit of the path π , then $[\![\pi]\!]^\sharp L$ is the set of variables which are live at the beginning of π
- → **Correctness of the transformation along a path:** If the value of a variable is accessed, this variable is necessarily live. The value of dead variables thus is irrelevant
- → **Correctness of the transformation:** In any execution of the transformed programs, the live variables always receive the same values

Computation of the sets $\mathcal{L}^*[u]$:

(1) Collecting constraints:

$$\begin{array}{lll} \mathcal{L}[stop] & \supseteq & \emptyset \\ \mathcal{L}[u] & \supseteq & [\![k]\!]^\sharp\,(\mathcal{L}[v]) \qquad k = (u, \_, v) \quad \text{edge} \end{array}$$

(2) Solving the constraint system by means of RR iteration.

Since $\mathbb{L}$ is finite, the iteration will terminate

(3) If the exit is (formally) reachable from every program point, then the least solution $\mathcal{L}$ of the constraint system equals $\mathcal{L}^*$ since all $[\![k]\!]^\sharp$ are distributive

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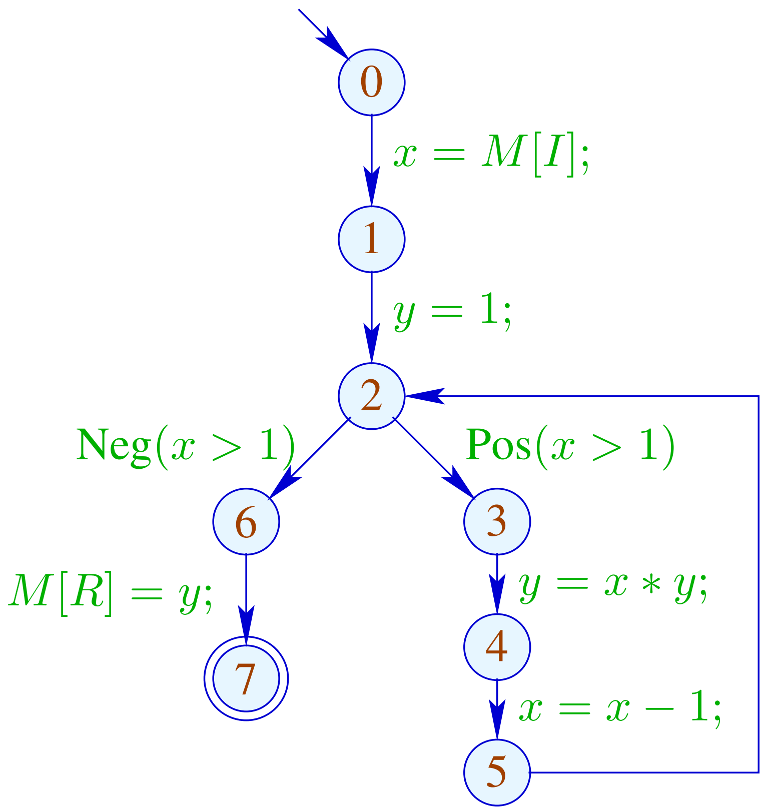
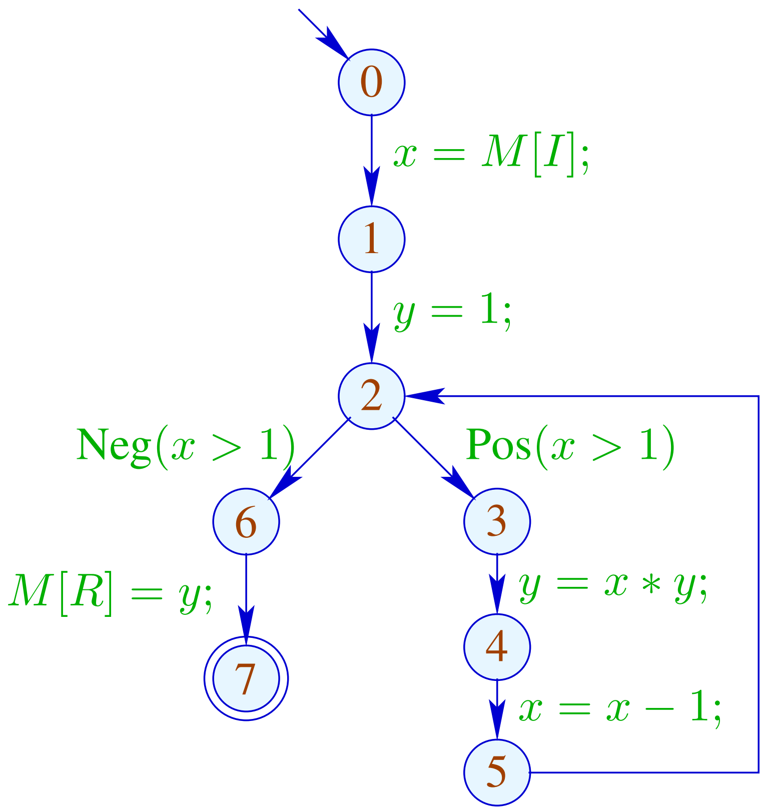
Note: The information is propagated backwards!

Example:



$$
\begin{array}{lcl}
\mathcal{L}[0] & \supseteq & (\mathcal{L}[1] \backslash \{x\}) \cup \{I\} \\
\mathcal{L}[1] & \supseteq & \mathcal{L}[2] \backslash \{y\} \\
\mathcal{L}[2] & \supseteq & (\mathcal{L}[6] \cup \{x\}) \cup (\mathcal{L}[3] \cup \{x\}) \\
\mathcal{L}[3] & \supseteq & (\mathcal{L}[4] \backslash \{y\}) \cup \{x, y\} \\
\mathcal{L}[4] & \supseteq & (\mathcal{L}[5] \backslash \{x\}) \cup \{x\} \\
\mathcal{L}[5] & \supseteq & \mathcal{L}[2] \\
\mathcal{L}[6] & \supseteq & \mathcal{L}[7] \cup \{y, R\} \\
\mathcal{L}[7] & \supseteq & \emptyset
\end{array}
$$

Example:

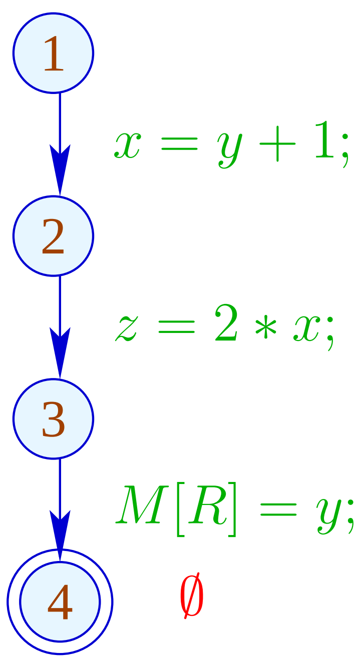


	1	2
7	$\emptyset$	
6	$\{y, R\}$	
2	$\{x, y, R\}$	dito
5	$\{x, y, R\}$	
4	$\{x, y, R\}$	
3	$\{x, y, R\}$	
1	$\{x, R\}$	
0	$\{I, R\}$	

The left-hand side of no assignment is dead

Caveat:

Removal of assignments to dead variables may kill further variables:



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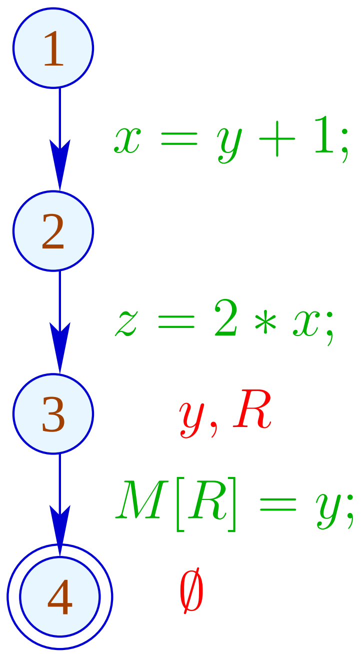

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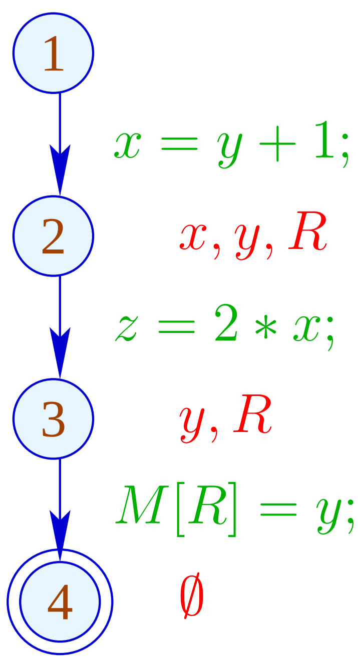

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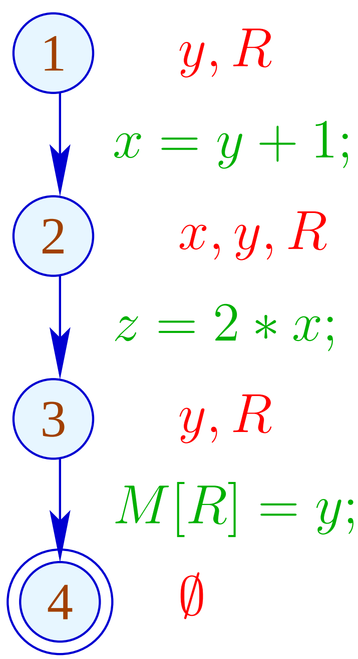

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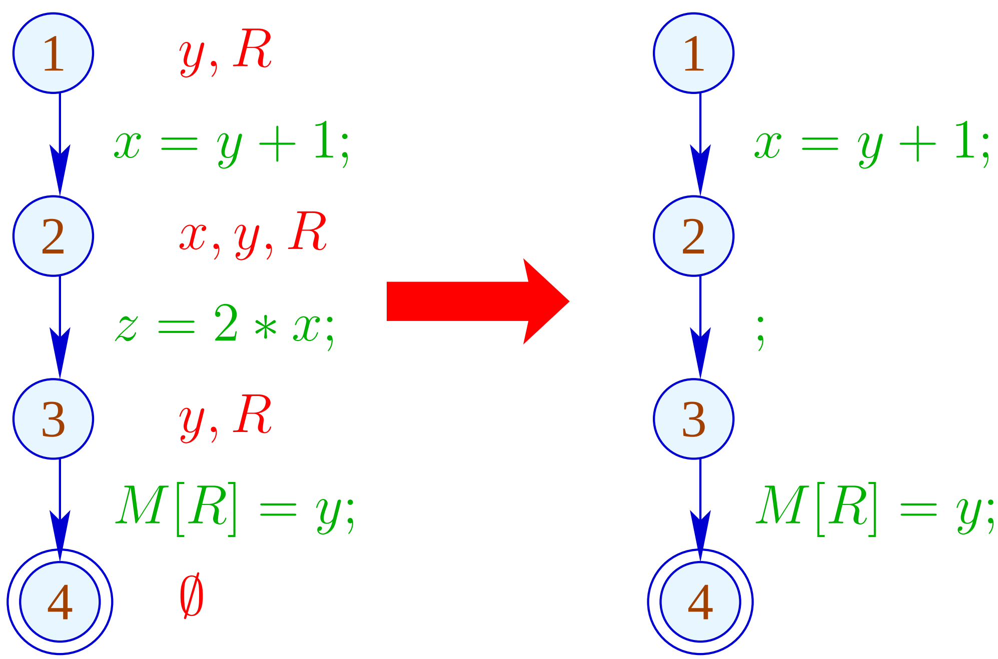

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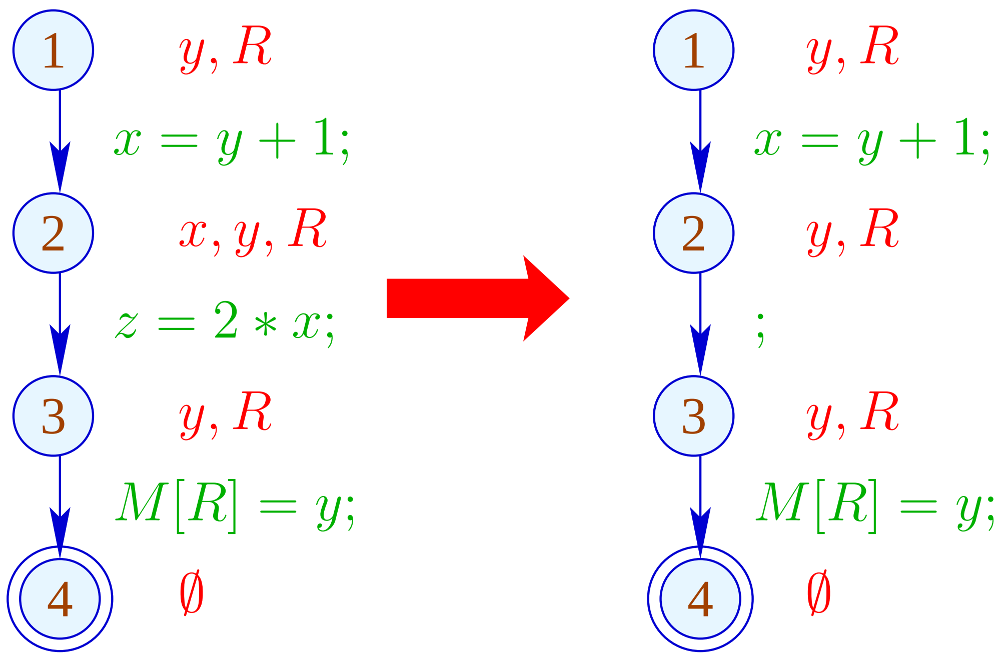

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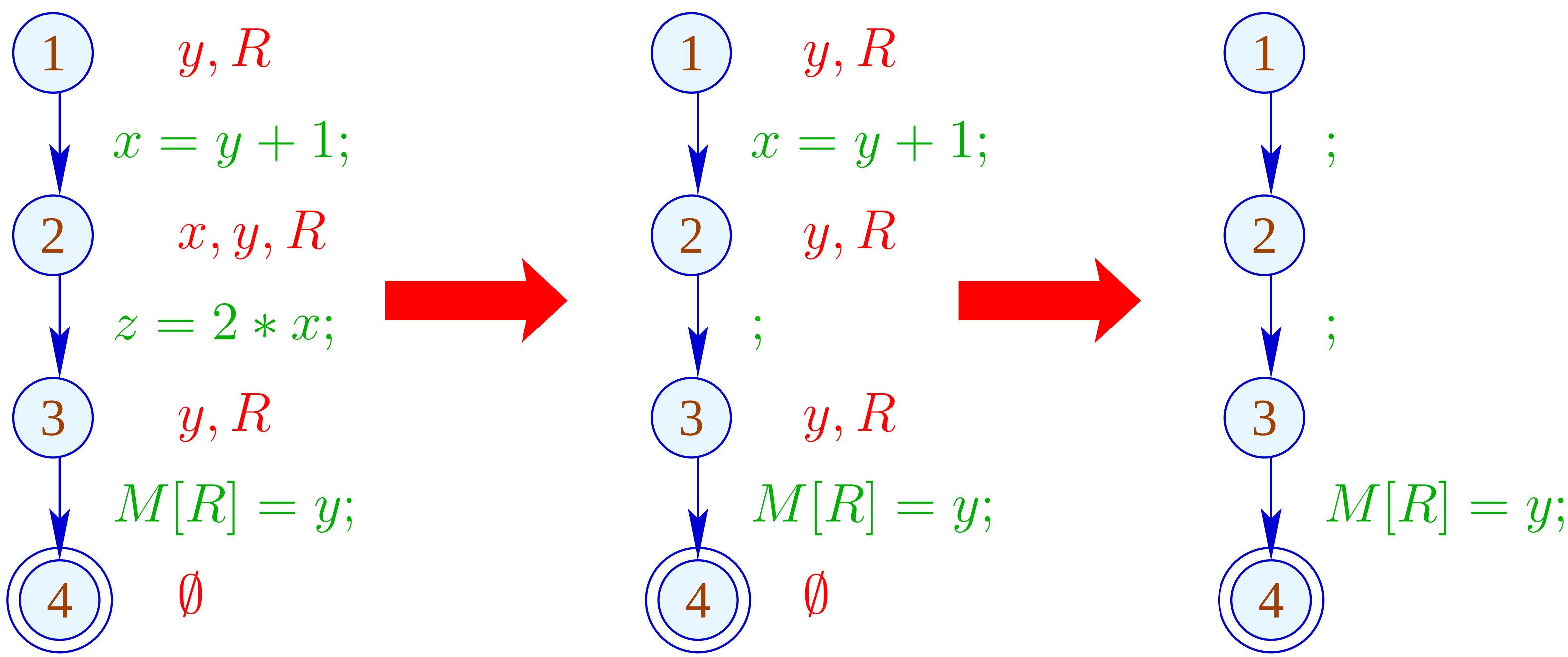


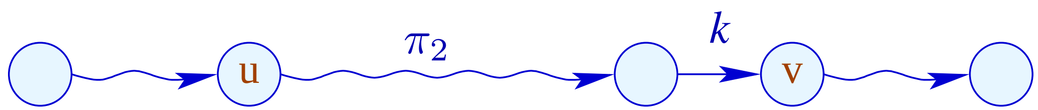
Re-analyzing the program is inconvenient

Idea: Analyze **true** liveness!

x is called **truly live** at u along a path π, either

if π can be decomposed into $\pi = \pi_1 \, k \, \pi_2$ such that:

- k is a **true** use of x ;
- π_1 does not contain any **definition** of x.

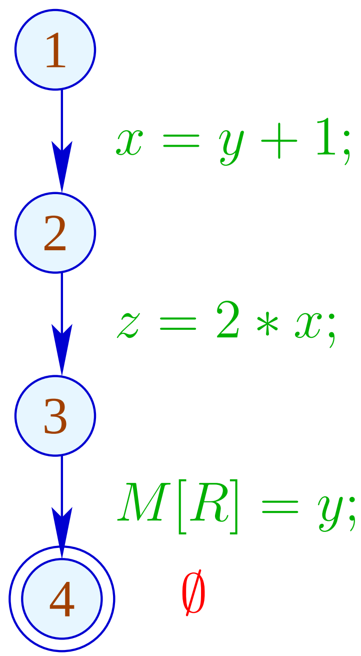


The set of truly used variables at an edge $k = (\_, lab, v)$ is defined as:

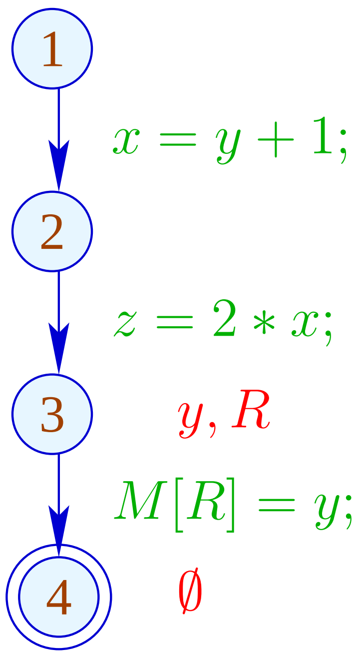
lab	truly used
$;$	$\emptyset$
$\text{true}\,(e)$	$Vars\,(e)$
$\text{false}\,(e)$	$Vars\,(e)$
$x = e;$	$Vars\,(e)$ $(*)$
$x = M[e];$	$Vars\,(e)$ $(*)$
$M[e_1] = e_2;$	$Vars(e_1) \cup Vars(e_2)$

$(*)$ – given that x is truly live at v

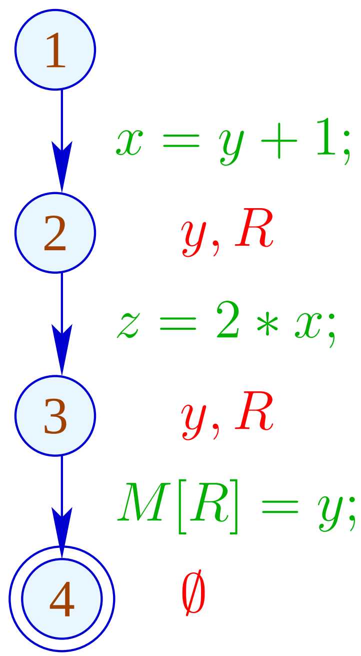
Example:



Example:



Example:



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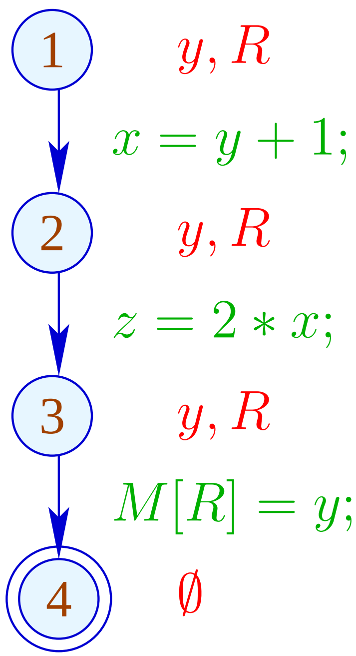

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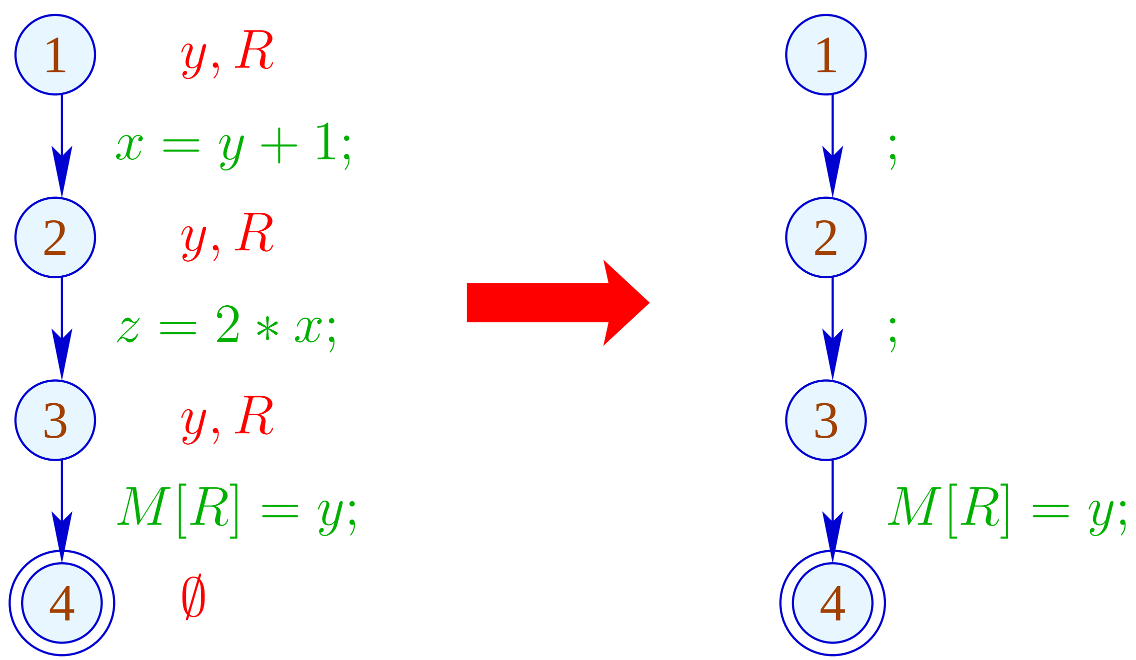

The Effects of Edges:

$$
\begin{array}{lcl}
[\![;]\!]^\sharp\, L & = & L \\
[\![\text{true}(e)]\!]^\sharp\, L & = & [\![\text{false}(e)]\!]^\sharp\, L \;\; = \;\; L \cup \mathit{Vars}(e) \\
[\![x = e;]\!]^\sharp\, L & = & (L \backslash \{x\}) \cup \qquad\qquad \mathit{Vars}(e) \\
[\![x = M[e];]\!]^\sharp\, L & = & (L \backslash \{x\}) \cup \qquad\qquad \mathit{Vars}(e) \\
[\![M[e_1] = e_2;]\!]^\sharp\, L & = & L \cup \mathit{Vars}(e_1) \cup \mathit{Vars}(e_2)
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  To see this, consider for $\mathbb{D} = 2^U$, $f\, y = (u \in y)\, ?\, b : \emptyset$ We verify:

$$\begin{aligned} f\,(y_1 \cup y_2) &= (u \in y_1 \cup y_2)\, ?\, b : \emptyset \\ &= (u \in y_1 \vee u \in y_2)\, ?\, b : \emptyset \\ &= (u \in y_1)\, ?\, b : \emptyset \cup (u \in y_2)\, ?\, b : \emptyset \\ &= f\, y_1 \cup f\, y_2 \end{aligned}$$

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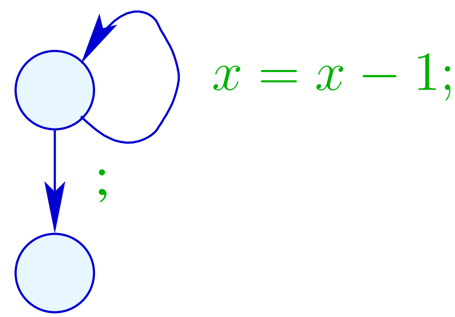
&= f\, y_1 \cup f\, y_2
\end{aligned}
$$

$\Longrightarrow$ the constraint system yields the MOP

- True liveness detects more superfluous assignments than repeated liveness !!!



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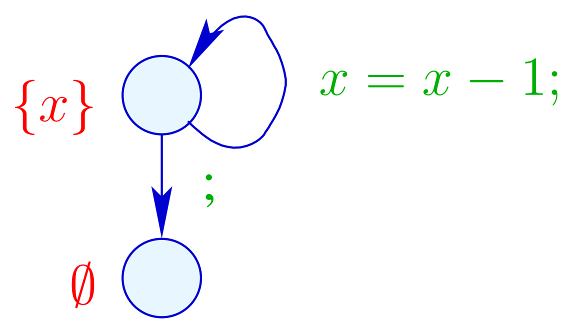
Liveness:



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True Liveness: